Chapter 08

Hypothesis Testing: Two-Sample Inference

Fundamentals of Biostatistics

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8.1 Introduction

All the tests introduced in Chapter 7 were one-sample tests, in which the underlying parameters of the population from which the sample was drawn were compared with comparable values from other generally large populations whose parameters were assumed to be known.

A more frequently encountered situation is the two-sample hypothesis-testing problem.

DEFINITION 8.1 In a **two-sample hypothesis-testing problem**, the underlying parameters of two different populations, *neither of whose values is assumed known*, are compared.

DEFINITION 8.4 Two samples are said to be **paired** when each data point in the first sample is matched and is related to a unique data point in the second sample.

DEFINITION 8.5 Two samples are said to be **independent** when the data points in one sample are unrelated to the data points in the second sample.

In this chapter (chapter 8), the appropriate methods of hypothesis testing for both the paired-sample and independent-sample situations are studied.

8.2 – 8.3 The Paired t Test and Interval Estimation

In this section, we discuss the estimation and hypothesis testing when the samples are dependent or related (paired samples). In this case, two data values x_{i1} and x_{i2} (*one in each sample*) for $i = 1, 2, \dots, n$ are collected from the same element (*unit or item*). Hence they are called paired or matched samples. Consider the difference:

$$d_i = x_{i2} - x_{i1} \; ; \; i = 1, 2, \dots, n$$

then the structure of the paired data takes the following form:

Element Number (i)	Sample 1	Sample 2	Difference (d_i)
1	x_{11}	x_{12}	$d_1 = x_{12} - x_{11}$
2	x_{21}	x_{22}	$d_2 = x_{22} - x_{21}$
.	.	.	
.	.	.	
.	.	.	
n	x_{n1}	x_{n2}	$d_n = x_{n2} - x_{n1}$

The differences $d_1, d_2, \dots, d_n$ represents a random sample of size n (*number of matched pairs*) with sample mean ($\bar{d}$) and sample standard deviation (S_d), where:

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} \text{ and } S_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} d_i^2 - n\,(\bar{d})^2}{n-1}} = \sqrt{\frac{\left[\sum_{i=1}^{n} d_i^2 - \frac{\left(\sum_{i=1}^{n} d_i\right)^2}{n}\right]}{n-1}}$$

Sampling Distribution of $\bar{d}$

Let d_1, d_2, ... , d_n be a random sample of size n from $N(\mu_d, \sigma_d^2)$, that is, d_i is normally distributed with mean μ_d and unknown variance by σ_d^2. Then the sampling distribution of the sample mean $(\bar{d})$ is approximately normal with the following mean and standard deviation:

$$\mu_{\bar{d}} = \mu_d \quad \text{and} \quad \sigma_{\bar{d}} = \sqrt{\sigma_{\bar{d}}^2} = \frac{\sigma_d}{\sqrt{n}}$$

where

μ_d = mean of the paired differences for the population

σ_d = standard deviation of the paired differences for the population

Usually the sample size (n) is small and standard deviation (σ_d) is unknown in the case of paired data. This leads to the following test statistic for the mean μ_d:

$$t = \frac{\bar{d} - \mu_d}{S_d / \sqrt{n}} \sim t - \text{distribution with degrees of freedom} = n - 1$$

where S_d = sample standard deviation of the paired differences for the sample.

Now, based on the sampling distribution of the sample mean $(\bar{d})$ the $(1-\alpha)100\%$ confidence interval (CI) for μ_d and the hypothesis testing using the one-sample t test procedure called the paired t test can be obtained as follows:

(I) Interval Estimation for μ_d

The two-sided $(1-\alpha)100\%$ confidence interval (CI) for the true mean difference (Δ) or μ_d can be constructed as follows:

$$\text{CI} = \bar{d} \pm t_{(n-1\,,1-(\alpha/2))} \frac{S_d}{\sqrt{n}}$$

(II) Statistical Test (*Paired t Test*) for μ_d

The hypotheses and the rejection regions at level of significance α can be described as follows:

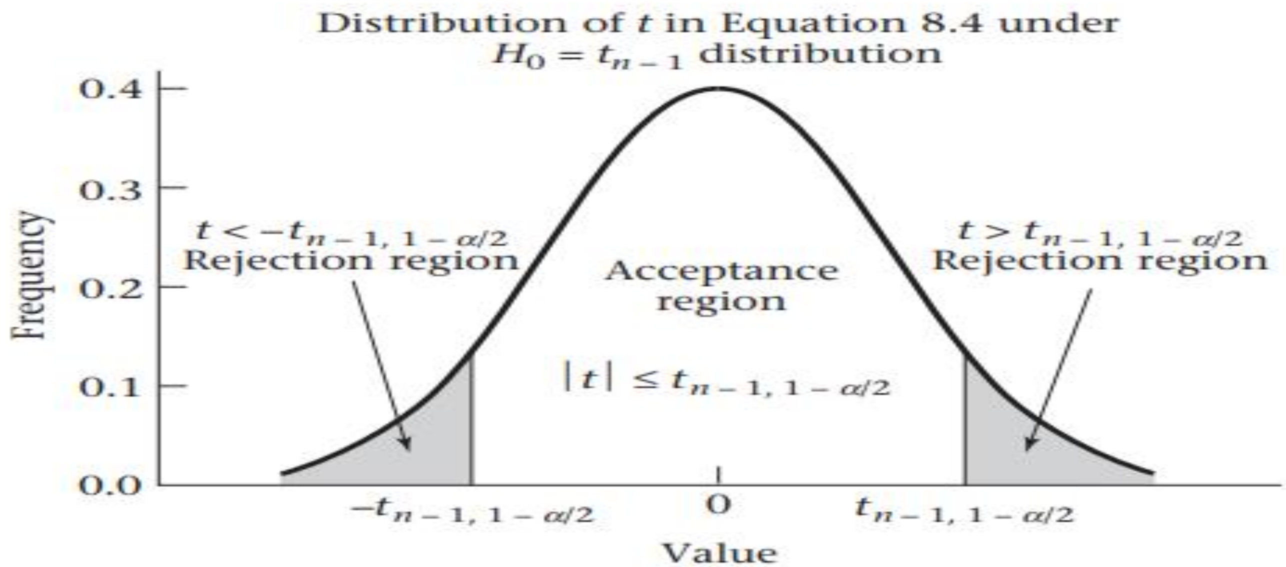
$H_0: \mu_d = 0 \;\; vs \;\; H_1: \mu_d > 0$ then reject H_0 if $t > t_{(n-1\,,1-\alpha)}$ otherwise Accept H_0.

$H_0: \mu_d = 0 \;\; vs \;\; H_1: \mu_d < 0$ then reject H_0 if $t < -t_{(n-1\,,1-\alpha)}$ otherwise Accept H_0.

$H_0: \mu_d = 0 \;\; vs \;\; H_1: \mu_d \neq 0$ then reject H_0 if $t > t_{(n-1\,,1-\alpha/2)}$ or $t < -t_{(n-1\,,1-\alpha/2)}$
Otherwise Accept H_0.

Note that, in cases when the sample size is large $(n \geq 30)$ the one-sample Z-test can be used to make the inferences about the mean μ_d.

FIGURE 8.1 Acceptance and rejection regions for the paired t test



p-value

The p-value for the two-sided paired t test can be computed as follows:

EQUATION 8.5

Computation of the *p*-Value for the Paired *t* Test

If $t < 0$,

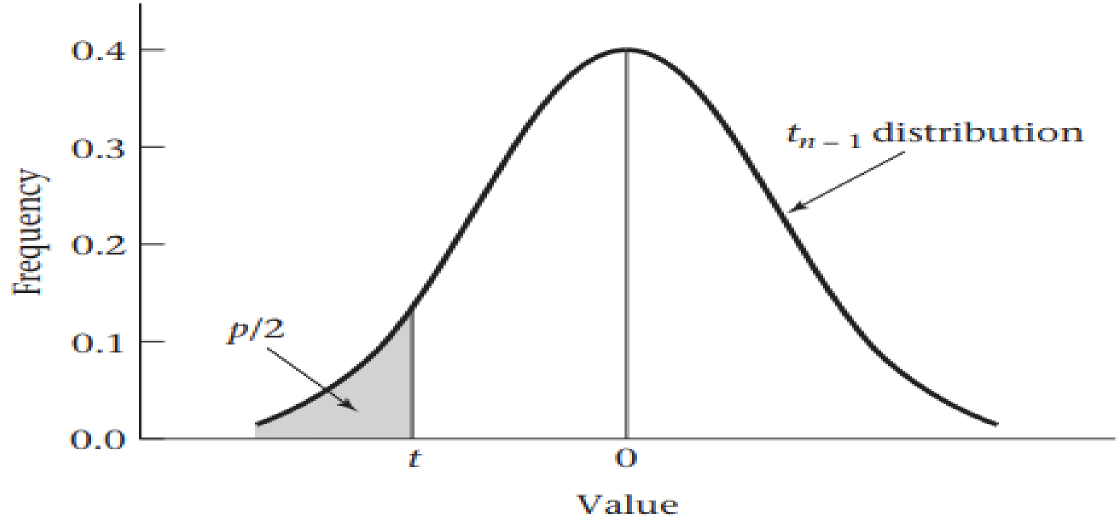
$$p = 2 \times [\text{the area to the left of } t = \bar{d}/(s_d/\sqrt{n}) \text{ under a } t_{n-1} \text{ distribution}]$$

If $t \ge 0$,

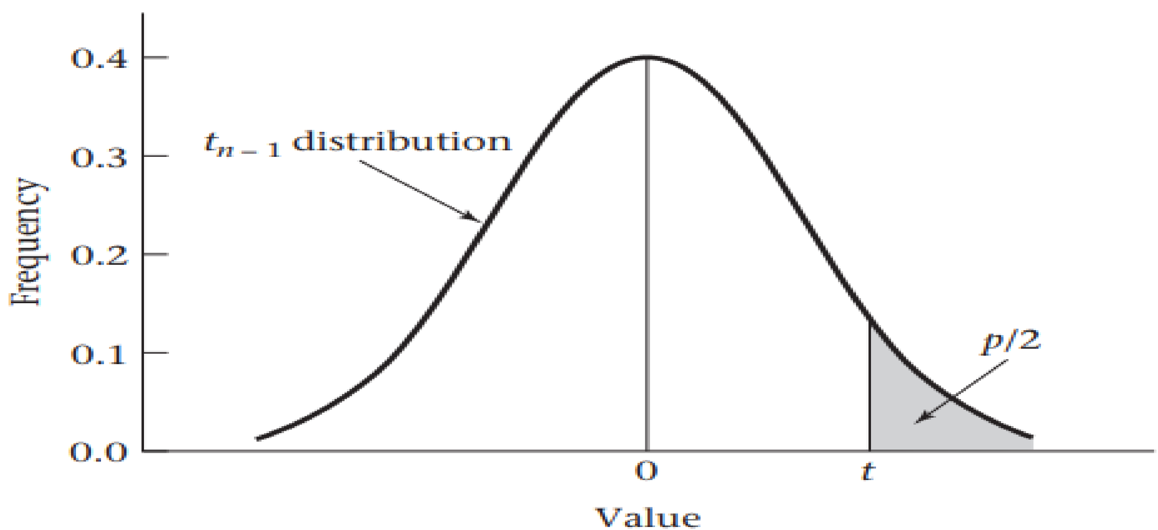
$$p = 2 \times [\text{the area to the right of } t \text{ under a } t_{n-1} \text{ distribution}]$$

The computation of the p-value is illustrated in Figure 8.2.

FIGURE 8.2 Computation of the *p*-value for the paired *t* test



If $t = \bar{d}/(s_d/\sqrt{n}) < 0$, then $p = 2 \times$ (area to the left of t under a t_{n-1} distribution).



If $t = \bar{d}/(s_d/\sqrt{n}) \geq 0$, then $p = 2 \times$ (area to the right of t under a t_{n-1} distribution).

EXAMPLE 8.2

Hypertension Let's say we are interested in the relationship between oral contraceptive (OC) use and blood pressure in women. The following experimental design can be used to assess this relationship:

(1) Identify a group of nonpregnant, premenopausal women of childbearing age (16–49 years) who are not currently OC users, and measure their blood pressure, which will be called the *baseline blood pressure*.

(2) Rescreen these women 1 year later to ascertain a subgroup who have remained nonpregnant throughout the year and have become OC users. This subgroup is the study population.

(3) Measure the blood pressure of the study population at the follow-up visit.

(4) Compare the baseline and follow-up blood pressure of the women in the study population to determine the difference between blood pressure levels of women when they *were* using the pill at follow-up and when they *were not* using the pill at baseline.

The above designed is the paired-sample study design. Suppose that the sample data in Table 8.1 are obtained. The systolic blood-pressure (SBP) level of the i^{th} woman is denoted at baseline by x_{i1} and at follow-up by x_{12}. Assume that the SBP of the i^{th} woman is normally distributed at baseline with mean μ_i and variance σ^2 and at follow-up with mean $\mu_i + \Delta$ (where $\mu_d = \Delta$) and variance σ^2.

We are thus assuming that the underlying mean difference in SBP between follow-up and baseline is Δ.

- If Δ = 0, then there is no difference between mean baseline and follow-up SBP.
- If Δ > 0, then using OC pills is associated with a raised mean SBP.
- If Δ < 0, then using OC pills is associated with a lowered mean SBP.

We want to test the hypothesis $\mathrm{H}_0: \mu_d = 0$ vs. $\mathrm{H}_1: \mu_d \neq 0$.

TABLE 8.1 SBP levels (mm Hg) in 10 women while not using (baseline) and while using (follow-up) OCs

i	SBP level while not using OCs (x_{i1})	SBP level while using OCs (x_{i2})
1	115	128
2	112	115
3	107	106
4	119	128
5	115	122
6	138	145
7	126	132
8	105	109
9	104	102
10	115	117

Answer the following:

(a) Using the data in Table 8.1, compute a 95% CI for the true mean SBP after starting OCs (mean of differences (μ_d))?

(b) Can you conclude that the of using of OC pills is effective in increasing the SBP levels (mm Hg)? Use $\alpha = 0.05 = 5\%$?

Solution

Element Number (i)	SBP level while not using OCs (x_{i1})	SBP level while not using OCs (x_{i2})	$d_i = x_{i2} - x_{i1}$	d_i^2
1	115	128	13	169
2	112	115	3	9
3	107	106	-1	1
4	119	128	9	81
5	115	122	7	49
6	138	145	7	49
7	126	132	6	36
8	105	109	4	16
9	104	102	-2	4
10	115	117	2	4
	Sum		**48**	**418**

The sample mean ($\bar{d}$) and the sample standard deviation (S_d) of the differences d_i's are computed to be as follows:

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{\sum_{i=1}^{10} d_i}{10} = \frac{13+3+ \ldots +2}{10} = \frac{48}{10} = 4.8$$

$$S_d = \sqrt{\frac{\left[\sum_{i=1}^{n} d_i^2 - \frac{\left(\sum_{i=1}^{n} d_i\right)^2}{n}\right]}{n-1}} = \sqrt{\frac{\left[418 - \frac{(48)^2}{10}\right]}{10-1}} = \sqrt{\frac{418-230.4}{9}} = 4.566$$

(a) The 95% CI for the mean of differences (μ_d) can be calculated as follows:

$$\text{CI} = \bar{d} \pm t_{(n-1\,,1-(\alpha/2))} \frac{S_d}{\sqrt{n}}$$

Step(1)

$(1-\alpha)100\% = 95\%$

$1 - \alpha = 0.95$

$\alpha = 0.05$

$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$

$1 - (\frac{\alpha}{2}) = 1 - 0.025 = 0.975$

$df = n - 1 = 10 - 1 = 9$

$t_{(n-1,\, 1-\frac{\alpha}{2})} = t_{(9\,,\, 0.975)} = 2.262$

From Table 5 in the Appendix

Step(2)

Lower Limit = $\bar{d} - t_{(n-1\,,1-(\alpha/2))}\frac{S_d}{\sqrt{n}}$

$= 4.8 - (2.262)\,(\frac{4.566}{\sqrt{10}})$

$= 4.8 - 3.266$

$= 1.534 \text{ mm Hg}$

Upper Limit = $\bar{d} + t_{(n-1\,,1-(\alpha/2))}\frac{S_d}{\sqrt{n}}$

$= 4.8 + (2.262)\,(\frac{4.566}{\sqrt{10}})$

$= 4.8 + 3.266$

$= 8.066 \text{ mm Hg}$

Conclusion: Then the 95% confidence interval for the true mean SBP change (μ_d) is CI = (L, U) = (1.534, 8.066) mm Hg. Thus, the true change in mean SBP is most likely between 1.5 and 8.1 mm Hg.

(b) Can you conclude that the of using of OC pills is effective in increasing the SBP levels (mm Hg)? Use $\alpha = 0.05 = 5\%$?

Step(1)

The null and alternative hypotheses H_0 and H_1 can be written as follows:

$$H_0: \mu_d = 0 \;\; vs \;\; H_1: \mu_d \neq 0$$

Step(2)

The value of the corresponding test statistic t is calculated as follows:

$$t = \frac{\bar{d} - \mu_d}{S_d/\sqrt{n}} = \frac{4.8 - 0}{\frac{4.566}{\sqrt{10}}} = 3.324$$

Step(3)

The critical-value can be obtained as follows:

$$t_{(n-1,1-\alpha/2)} = t_{(9,0.975)} = 2.262$$

Step(4)

The rejection rule is given as follows: Rreject H_0 at level of significance α if

$$t > t_{(n-1,1-\alpha/2)} \text{ or } t < -t_{(n-1,1-\alpha/2)}$$

Otherwise Accept H_0 ($|t| \leq t_{(n-1,1-\alpha/2)}$).

Step(5)

We get $t = 3.324 > t_{(9,0.975)} = 2.262$

Step(6)

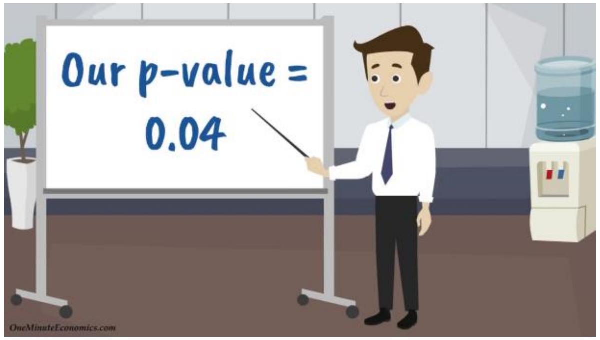
Decision: It follows that H_0 can be rejected (*there is a difference between mean baseline and follow-up SBP*) using a two-sided paired t test at α = .05.

Conclusion: There is a relationship between oral contraceptive (OC) use and blood pressure in women. We conclude that the using of OC pills is effective in increasing the SBP levels (mm Hg).

To compute an approximate p-value, we use the formula given in Equation 8.5 and then refer to Table 5 in the Appendix as follows:

If $t = 3.324 > 0$, then the p-value can be calculated using the following formula:

$$p = 2 \times [\text{the area to the right of } t \text{ under a } t_{(n-1)} \text{ distribution}]$$
$$= 2 \times P(t_{(n-1)} > t)$$
$$= 2 \times [1 - P(t_{(n-1)} \leq t)]$$
$$= 2 \times [1 - P(t_9 \leq 3.324)]$$
$$= 2 \times [1 - 0.995]$$
$$= 2 \times [0.005]$$
$$= 0.01$$



Now by using the p-value method we have:

$$p = 0.01 < \alpha = 0.05$$

then it follows that H_0 can be rejected using a two-sided Significance paired t test with $\alpha = 0.05$.

Notation

To compute a more exact p-value, a computer program like Minitab must be used.

8.4 Two-Sample t Test for Independent Samples (Equal Variances)

Suppose that we have two populations which are normally distributed. If the first population has a mean μ_1 and a variance σ_1^2 (*or a standard deviation* $\sigma_1 = \sqrt{\sigma_1^2}$), and the second population has a mean μ_2 and a variance σ_2^2 (*or a standard deviation* $\sigma_2 = \sqrt{\sigma_2^2}$). Also, suppose that two independent random samples (*groups*) are drawn from these two populations. The first sample of size n_1 is drawn from the first population and has a sample mean ($\overline{X}_1$) and a sample variance (S_1^2). The second sample of size n_2 is drawn from the second population and has a sample mean ($\overline{X}_2$) and a sample variance (S_2^2). We want to test the hypothesis:

$$H_0: \mu_1 = \mu_2 \ \ vs \ \ H_1: \mu_1 \neq \mu_2$$

Assume that the underlying variances in the two populations are the same or equal (that is, $\sigma_1^2 = \sigma_2^2 = \sigma^2$). We know that $\overline{X}_1$ is normally distributed with mean μ_1 and variance σ^2/n_1 and $\overline{X}_2$ is normally distributed with mean μ_2 and variance σ^2/n_2.

It seems reasonable to base the significance test on the difference between the two sample means, $\overline{X}_1 - \overline{X}_2$ which is normally distributed with mean μ_1- μ_2 and variance $\sigma^2(1/n_1 + 1/n_2)$. In symbols, as follows:

EQUATION 8.7

$$\bar{X}_1 - \bar{X}_2 \sim N\left[\mu_1 - \mu_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right]$$

Under H_0, we know that $\mu_1 - \mu_2$. Thus, Equation 8.7 reduces to

EQUATION 8.8

$$\bar{X}_1 - \bar{X}_2 \sim N\left[0, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right]$$

If σ^2 were known, then $\bar{X}_1 - \bar{X}_2$ could be divided by $\sigma\sqrt{(1/n_1 + 1/n_2)}$. From Equation 8.8, we have:

EQUATION 8.9

$$\frac{\bar{X}_1 - \bar{X}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

The test statistic in Equation 8.9 could be used as a basis for the hypothesis test. Unfortunately, σ^2 in general is unknown and must be estimated from the data. The best estimate of the population variance σ^2, which is denoted by S^2, is given by a weighted average of the two sample variances, where the weights are the number of df in each sample. In particular, S^2 will then have $(n_1 - 1)\ df$ from the first sample and $(n_2 - 1)\ df$ from the second sample, or:

$$[(n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2]\ df \text{ overall.}$$

EQUATION 8.10

The **pooled estimate of the variance** from two independent samples is given by

$$s^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$$

where $s = \sqrt{\left[(n_1-1)s_1^2 + (n_2-1)s_2^2\right]/(n_1+n_2-2)}$

Then S can be substituted for σ in Equation 8.9, and the resulting test statistic can then be shown to follow a t distribution with $n_1 + n_2 - 2\ df$ rather than a standard normal distribution, N(0, 1), distribution because σ^2 is unknown. Thus, the following test procedure is used:

EQUATION 8.11

Two-Sample *t* Test for Independent Samples with Equal Variances

Suppose we wish to test the hypothesis $H_0: \mu_1 = \mu_2$ vs. $H_1: \mu_1 \neq \mu_2$ with a significance level of α for two normally distributed populations, where σ^2 is assumed to be the same for each population.

Compute the test statistic:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

where $s = \sqrt{\left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right]/(n_1 + n_2 - 2)}$

If $t > t_{n_1+n_2-2,1-\alpha/2}$ or $t < -t_{n_1+n_2-2,1-\alpha/2}$

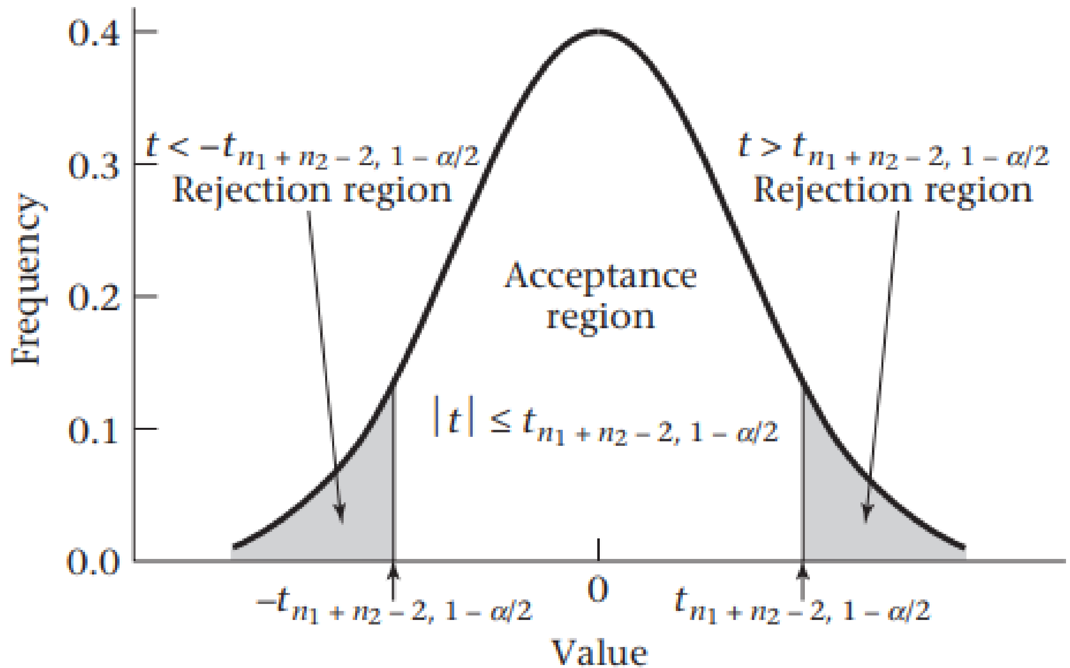
then H_0 is rejected.

If $-t_{n_1+n_2-2,1-\alpha/2}$ or $t \le t_{n_1+n_2-2,1-\alpha/2}$

then H_0 is accepted.

The acceptance and rejection regions for this test are shown in Figure 8.3.

FIGURE 8.3 Acceptance and rejection regions for the two-sample *t* test for independent samples with equal variances



Distribution of *t* in Equation 8.11 under $H_0 = t_{n_1+n_2-2}$ distribution

Similarly, a p-value can be computed for the test. Computation of the p-value depends on whether $\overline{X}_1 \le \overline{X}_2$ $(t \le 0)$ or $\overline{X}_1 > \overline{X}_2$ $(t > 0)$. In each case, the p-value corresponds to the probability of obtaining a test statistic at least as extreme as the observed value t. This is given in Equation 8.12.

EQUATION 8.12

Computation of the p-Value for the Two-Sample *t* Test for Independent Samples with Equal Variances

Compute the test statistic:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

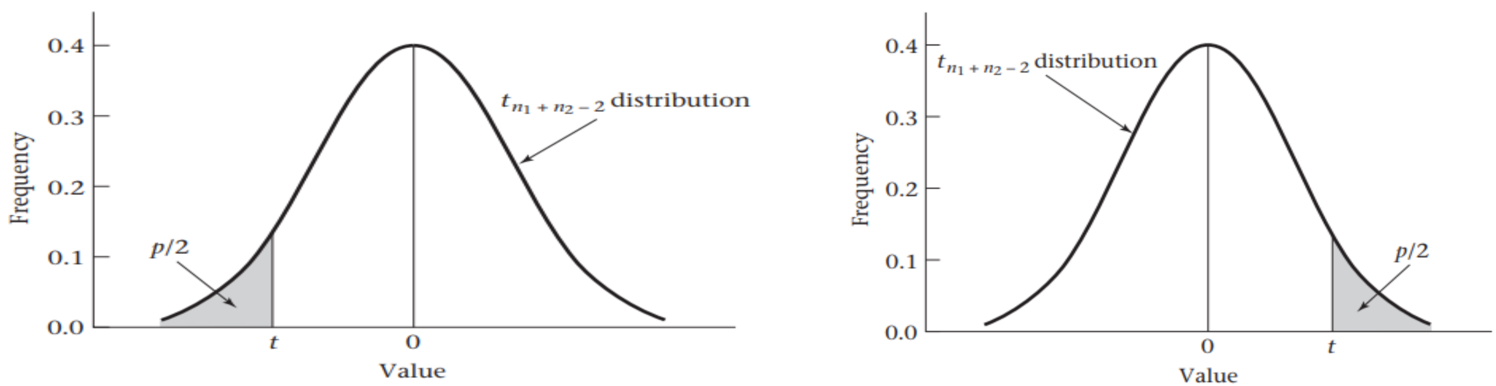
where $s = \sqrt{\left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right]/(n_1 + n_2 - 2)}$

If $t \le 0$, $p = 2 \times$ (area to the left of t under a $t_{n_1+n_2-2}$ distribution).

If $t > 0$, $p = 2 \times$ (area to the right of t under a $t_{n_1+n_2-2}$ distribution).

The computation of the p-value is illustrated in Figure 8.4.

FIGURE 8.4 **Computation of the *p*-value for the two-sample *t* test for independent samples with equal variances**



If $t = (\bar{x}_1 - \bar{x}_2)\Big/\left(s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) \le 0$, then $p = 2 \times$ (area to the left of t under a $t_{n_1+n_2-2}$ distribution).

If $t = (\bar{x}_1 - \bar{x}_2)\Big/\left(s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) > 0$, then $p = 2 \times$ (area to the right of t under a $t_{n_1+n_2-2}$ distribution).

EXAMPLE 8.10

Hypertension Suppose a sample of eight 35- to 39-year-old nonprenant, premenopausal OC users who work in a company and have a mean systolic blood pressure (SBP) of 132.86 mm Hg and sample standard deviation of 15.34 mm Hg are identified. A sample of 21 nonpregnant, premenopausal, non-OC users in the same age group are similarly identified who have mean SBP of 127.44 mm Hg and sample standard deviation of 18.23 mm Hg. What can be said about the underlying mean difference (μ_1 - μ_2) in blood pressure between the two groups? Assess the statistical significance of the data using $\alpha = 0.05$?

Solution

Step (1)

Sample Number	Sample Size	Sample Mean	Sample Standard Deviation
Sample 1	$n_1 = 8$	$\overline{X}_1 = 132.86$	$S_1 = 15.34$
Sample 2	$n_2 = 21$	$\overline{X}_2 = 127.44$	$S_2 = 18.23$

Step (2) Define the two population means (μ_1) and (μ_2) as follows:

μ_1 = The mean blood pressures of the OC users.

μ_2 = The mean blood pressures of the non-OC users.

We want to test using $\alpha = 0.05$ the hypothesis:

$$H_0: \mu_1 = \mu_2 \ \ vs \ \ H_1: \mu_1 \neq \mu_2$$

Step (3)

The pooled estimate of the sample standard deviation (S) from the two independent samples is calculated as follows:

$$S = \sqrt{S^2} = \sqrt{\frac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{(8-1)(15.34)^2 + (21-1)(18.23)^2}{8 + 21 - 2}}$$

$$= \sqrt{\frac{1647.2092 + 6646.658}{27}} = \sqrt{\frac{8293.8672}{27}} = 17.527$$

Step (4)

The t-test statistic can be calculated as follows:

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S * \sqrt{(1/n_1 + 1/n_2)}} = \frac{(132.86 - 127.44) - 0}{(17.527)\left(\sqrt{\frac{1}{8} + \frac{1}{21}}\right)} = \frac{5.42}{7.282} = 0.744 > 0$$

Step (5)

Since we have $t = 0.744 > 0$, then the rejection rule at level of significance α will be as follows:

$$Rule = \begin{cases} \text{Reject } H_0 \text{ if } t > t_{(n_1 + n_2 - 2,\, 1-(\alpha/2))} \\ \text{Accept } H_0 \text{ if } t \le t_{(n_1 + n_2 - 2,\, 1-(\alpha/2))} \end{cases}$$

Step (7)

The critical value is obtained from Table 5 in the Appendix as follows:

$$t_{(n_1 + n_2 - 2\,,\,1-(\alpha/2))}$$
$$= t_{(8 + 21 - 2\,,\,1-(0.05/2))}$$
$$= t_{(27,\ 0.975)}$$
$$= 2.052$$

Step (8)

The decision will be as follows:

We get

$$t = 0.744 < t_{(27,\ 0.975)} = 2.052$$

it follows that H_0 is accepted using a two-sided t-test at the $\alpha = 5\%$ level.

Conclusion

We conclude that the mean blood pressures of the OC users (μ_1) and the mean blood pressures of the non-OC users (μ_2) do not significantly differ from each other, that is, $\mu_1 = \mu_2$ or $\mu_1 - \mu_2 = 0$.

p-value

To compute an approximate p-value, and because we have $t = 0.744 > 0$, then we will use the following rule:

$$
\begin{aligned}
p &= 2 \times [\text{the area to the right of } t \text{ under a } t_{(n_1 + n_2 - 2)} \text{ distribution}] \\
&= 2 \times P(t_{(n_1 + n_2 - 2)} > t) \\
&= 2 \times [1 - P(t_{(n_1 + n_2 - 2)} \le t)] \\
&= 2 \times [1 - P(t_{27} \le 0.744)] \\
&= 2 \times [1 - 0.75] \\
&= 2 \times [0.25] \\
&= 0.50
\end{aligned}
$$

Now by using the p-value method we have:

$$p = 0.50 > \alpha = 0.05$$

then it follows that H_0 can be accepted using a two-sided Significance t test with α = 0.05.

Notation

The exact p-value obtained from MINITAB program is:

$$
\begin{aligned}
p &= 2 \times P(t_{27} > 0.744) \\
&= 0.46.
\end{aligned}
$$

8.5 Interval Estimation for the Comparison of Means from Two Independent Samples (Equal Variance Case)

In the previous section, methods of hypothesis testing for the comparison of means from two independent samples were discussed. It is also useful to compute the (1 − α) × 100% confidence intervals (CIs) for the true mean difference between the two groups (*or populations*) (μ1 − μ2) as follows:

EQUATION 8.13

Confidence Interval for the Underlying Mean Difference $(\mu_1 - \mu_2)$ Between Two Groups (Two-Sided) $(\sigma_1^2 = \sigma_2^2)$

A two-sided $100\% \times (1-\alpha)$ CI for the true mean difference $\mu_1 - \mu_2$ based on two independent samples with equal variance is given by

$$\left(\bar{x}_1 - \bar{x}_2 - t_{n_1+n_2-2,1-\alpha/2}\, s\sqrt{\frac{1}{n_1}+\frac{1}{n_2}},\ \bar{x}_1 - \bar{x}_2 + t_{n_1+n_2-2,1-\alpha/2}\, s\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)$$

where s^2 = pooled variance estimate given in Equation 8.12.

The derivation of this formula is provided in Section 8.11.

EXAMPLE 8.11

Hypertension Using the data in Examples 8.10, compute a 95% confidence interval (CI) for the true mean difference in systolic blood pressure (SBP) between 35- to 39-year-old OC users and non-OC users (μ_1 - μ_2)?

Solution

A confidence interval (CI) for the underlying mean difference (μ_1 - μ_2) in SBP between the population of 35- to 39-year-old OC users and non-OC users can be calculated as follows:

Step (1)

$$(1 - \alpha) \times 100\% = 95\%$$
$$1 - \alpha = 0.95$$
$$\alpha = 0.05$$
$$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$$
$$1 - \left(\frac{\alpha}{2}\right) = 1 - 0.95 = 0.975$$

Step (2)

The critical value is obtained from Table 5 in the Appendix as follows:

$$t_{\left(n_1 + n_2 - 2\ ,\ 1 - (\alpha/2)\right)}$$
$$= t_{(27,\ 0.975)}$$
$$= 2.052$$

Step (3)

Using Equation 8.13, the lower and upper limits for the (1 − α) × 100% = 95% confidence interval (CI) for the true mean difference in systolic blood pressure (SBP) between 35- to 39-year-old OC users and non-OC users (μ_1 - μ_2) can be calculated as follows:

$$CI = \left(\bar{x}_1 - \bar{x}_2 - t_{n_1+n_2-2,1-\alpha/2} s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}},\ \bar{x}_1 - \bar{x}_2 + t_{n_1+n_2-2,1-\alpha/2} s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

$$\text{Lower Limit} = (\bar{X}_1 - \bar{X}_2) - t_{(n_1 + n_2 - 2\ ,\ 1-(\alpha/2))}\ S\sqrt{(1/n_1 + 1/n_2)}$$

$$= (132.86 - 127.44) - \left[(2.052)(17.527)\left(\sqrt{\frac{1}{8} + \frac{1}{21}}\right)\right]$$

$$= 5.42 - 14.94$$

$$= -9.52$$

$$\text{Upper Limit} = (\bar{X}_1 - \bar{X}_2) + t_{(n_1 + n_2 - 2\ ,\ 1-(\alpha/2))}\ S\sqrt{(1/n_1 + 1/n_2)}$$

$$= (132.86 + 127.44) + \left[(2.052)(17.527)\left(\sqrt{\frac{1}{8} + \frac{1}{21}}\right)\right]$$

$$= 5.42 + 14.94$$

$$= 20.36$$

Conclusion: CI =(-9.52, 20.36)

We are 95% confident that the true mean difference (μ_1 - μ_2) in SBP between the population of 35- to 39-year-old OC users and non-OC users is between -9.52 and 20.36. This interval is rather wide and indicates that a much larger sample is needed to accurately assess the true mean difference.

Notation

In this section, we have introduced the two-sample t test for independent samples with equal variances. This test is used to compare the mean of a normally distributed random variable (or a random variable with samples large enough so that the central-limit theorem can be assumed to hold) between two independent samples with equal variances.

Problems

8.2 – 8.6, 8.15 – 8.18, 8.44 – 8.45