Chapter 07

Hypothesis Testing: One-Sample Inference

Fundamentals of Biostatistics

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7.1 Introduction

In chapter 6 we discuss the methods of point and interval estimation for population mean (μ) and population proportion (p) parameters of various distributions. In this chapter (chapter 7), some of the basic concepts of hypothesis testing are developed and applied to one-sample problems of statistical inference. In a one-sample problem, hypotheses are specified about a single distribution; in a two-sample problem, two different distributions are compared.

Question: Why is hypothesis testing so important?
Answer: Hypothesis testing provides an objective framework for making decisions using probabilistic methods, rather than relying on subjective impressions.

Notation: In terms of hypothesis testing, in particular, the hypotheses being considered can be formulated in terms of null and alternative hypotheses.

7.2 General Concepts

DEFINITION 7.1

The **null hypothesis,** denoted by H_0, is the hypothesis that is to be tested. The alternative hypothesis, denoted by H_1, is the hypothesis that in some sense contradicts the null hypothesis.

Notation

We will assume the underlying distribution is normal under either hypothesis.

Example

- The null hypothesis (H_0) is that the mean birthweight in the low-SES-area hospital (μ) is equal to the mean birthweight in the United States (μ_0).

- The alternative hypothesis (H_1) is that the mean birthweight in this the low-SES-area hospital (μ) is lower than the mean birthweight in the United States (μ_0).

Note that: SES means socioeconomic status.

The above two hypotheses can be written more succinctly in the following form:

versus

EQUATION 7.1

$$H_0: \mu = \mu_0 \quad \text{vs.} \quad H_1: \mu < \mu_0$$

Notations

- If we decide H_0 is true, then we say we accept H_0. If we decide H_1 is true, then we state that H_1 is not true or, equivalently, that we reject H_0.
- In actual practice, it is impossible, to prove that the null hypothesis is true. Thus, in particular, if we accept H_0, then we have actually failed to reject H_0.

DEFINITION 7.2 The probability of a **type I error** is the probability of rejecting the null hypothesis when H_0 is true.

Example

Probability of deciding that the mean birthweight in the hospital was lower than 120 oz $(\mu < 120)$ when in fact it was 120 oz $(\mu = 120)$. → **Type I Error**

DEFINITION 7.3 The probability of a **type II error** is the probability of accepting the null hypothesis when H_1 is true. This probability is a function of μ as well as other factors.

Example

Probability of deciding that the mean birthweight in the hospital was lower than 120 oz $(\mu = 120)$ when in fact it was 120 oz $(\mu < 120)$. → **Type II Error**

Notation: Type I and Type II Errors often result in monetary and nonmonetary costs. See Example 7.6 page 213.

Notation: The Greek letters α and β represent the probabilities of Type I and Type II Errors.

DEFINITION 7.4 The probability of a **type I error** is usually denoted by α and is commonly referred to as the **significance level** of a test.

α = Probability of a Type I Error
= **P(Type I Error)**
= P(Rejecting H_0 when H_0 is true)
→ Rejecting a Good H_0

DEFINITION 7.5 The probability of a **type II error** is usually denoted by β.

β = Probability of a Type II Error
= **P(Type II Error)**
= P(Not Rejecting H_0 when H_0 is false)
→ Accepting a Bad H_0

DEFINITION 7.6 The power of a test is defined as

$$1 - \beta = 1 - \text{probability of a type II error} = Pr(\text{rejecting } H_0 | H_1 \text{ true})$$

Notation: α and β should be as small as possible because they are probabilities of errors.

EXAMPLE 7.7

Rheumatology Suppose a new drug for pain relief is to be tested among patients with osteoarthritis (OA). The measure of pain relief will be the percent change in pain level as reported by the patient after taking the medication for 1 month. A random sample of size 50 OA patients will participate in the study. Then:

- What hypotheses are to be tested?
- What do type I error, type II error, and power mean in this situation?

Solution: We have:

μ = The mean decline in level of pain as measured by a pain relief scale over a 1-month period.

- A positive value for μ indicates improvement.
- A negative value indicates decline.

The hypotheses to be tested are:

$$H_0: \mu = 0 \ \ \textit{versus} \ \ H_1: \mu > 0$$

A type I Error: P(Deciding that the drug is an effective pain reliever based on data from 50 patients *when* the drug has no effect on pain relief).

A type II Error: P(Deciding that the drug has no effect on pain relief based on data from 50 patients *when* the drug is an effective pain reliever).

Power of the Test: P(Deciding that the drug is effective as a pain reliever based on data from 50 patients *when* it is effective).

Important Result

The general aim in hypothesis testing is to use statistical tests that make α and β as small as possible. This goal requires compromise because making α small involves rejecting the null hypothesis less often, whereas making β small involves accepting the null hypothesis less often. These actions are contradictory; that is, as α decreases, β increases, and as α increases, β decreases. Our general strategy is to fix α at some specific level (for example, 0.10, 0.05, 0.01, . . .) and to use the test that minimizes β or, equivalently, maximizes the power (1- β).

7.3 One-Sample Test for the Mean of a Normal Distribution (One-Sided Alternatives)

DEFINITION 7.7 The **acceptance region** is the range of values of $\bar{x}$ for which H_0 is accepted.

DEFINITION 7.8 The **rejection region** is the range of values of $\bar{x}$ for which H_0 is rejected.

DEFINITION 7.9 A **one-tailed test** is a test in which the values of the parameter being studied (in this case μ) under the alternative hypothesis are allowed to be either greater than or less than the values of the parameter under the null hypothesis (μ_0), *but not both*.

One-Tailed (Sided) Test Types

$H_0: \mu = \mu_0$ $vs.$ $H_1: \mu < \mu_0$ [lower-tailed (left-tailed) test]
$H_0: \mu = \mu_0$ $vs.$ $H_1: \mu > \mu_0$ [upper-tailed (right-tailed) test]

First Test Procedure

EQUATION 7.2

One-Sample *t* Test for the Mean of a Normal Distribution with Unknown Variance (Alternative Mean < Null Mean)

To test the hypothesis H_0: $\mu = \mu_0$, σ unknown vs. H_1: $\mu < \mu_0$, σ unknown with a significance level of α, we compute

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

→ Test Statistic

If $t < -t_{(n-1, 1-\alpha)}$ then we reject H_0.

If $t \geq -t_{(n-1, 1-\alpha)}$ then we do not reject (accept) H_0.

→ Critical Value

Notation: The conditions to use this Test Procedure are:
(1) Normal Distribution.
(2) Population Standard Deviation $\sigma = \sqrt{\sigma^2}$ is Unknown.
(3) The Sample size (n) is small ($n < 30$).

DEFINITION 7.12 The general approach in which we compute a test statistic and determine the outcome of a test by comparing the test statistic with a critical value determined by the type I error is called the **critical-value method** of hypothesis testing.

EXAMPLE 7.10

Obstetrics Use the one-sample t test to test the hypothesis:

$$H_0: \mu = 120 \ \ vs \ \ H_1: \mu < 120$$

based on the following information for a birthweight data which is normally distributed: $n = 25$, $\bar{X} = 115$ oz , $S = 24$ oz

using a significance level of $\alpha = 0.05$?

Solution

- Conditions: *We have:*
  1- Normal distribution (Normal population).
  2- The standard deviation σ is unknown (S = 24).
  3- The sample size (n) is small (n = 25 < 30).
- Test Statistic (Calculated Value)

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{115 - 120}{24/\sqrt{25}} = -1.042$$

- Critical Value(Tabulated Value)

$t_{(n-1,\, 1-\alpha)} = t_{(24,\, 0.95)} = 1.711$

which implies that

$-t_{(n-1,\, 1-\alpha)} = -1.711$

- Decision

We get

$t = -1.042 \ > \ -t_{(n-1,\, 1-\alpha)} = -1.711$

- Conclusion

We accept H_0 at significance level $\alpha = 0.05$ that is $\mu = 120$ oz.

Second Test Procedure

EQUATION 7.6

One-Sample t Test for the Mean of a Normal Distribution with Unknown Variance (Alternative Mean > Null Mean)

To test the hypothesis

$$H_0: \mu = \mu_0 \quad \text{vs.} \quad H_1: \mu > \mu_0$$

with a significance level of α, the best test is based on t, where

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

If $t > t_{(n-1,\, 1-\alpha)}$ then we reject H_0.

If $t \le t_{(n-1,\, 1-\alpha)}$ then we do not reject (accept) H_0.

Notation: The conditions to use this Test Procedure are:

(1) Normal Distribution.

(2) Population Standard Deviation $\sigma = \sqrt{\sigma^2}$ is Unknown.

(3) The Sample size (n) is small ($n < 30$).

EXAMPLE 7.18

Cardiovascular Disease, Pediatrics Suppose the mean cholesterol level of 10 children whose fathers died from heart disease is 210 mg/dL and the sample standard deviation is 50 mg/dL. Test the hypothesis that the mean cholesterol level (μ) is higher than $\mu_0 = 175$ mg/dl:

$$H_0\colon \mu = 175 \;\; vs \;\; H_1\colon \mu > 175$$

Assuming that the cholesterol level is normally distributed, use the one-sample t test at a significance level of $\alpha = 0.05$?

Solution

➢ Conditions: *We have:*

1- Normal distribution (Normal population).

2- The standard deviation σ is unknown (S = 50).

3- The sample size (n) is small (n = 10 < 30).

➢ Test Statistic (Calculated Value)

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{210 - 175}{50/\sqrt{10}} = 2.214$$

➢ Critical Value(Tabulated Value)

$$t_{(n-1,\,1-\alpha)} = t_{(9,\,0.95)} = 1.833$$

➢ Decision

We get

$$t = 2.214 \; > \; t_{(n-1,\,1-\alpha)} = 1.833$$

➢ Conclusion

We reject H_0 and accept H_1 at significance level $\alpha = 0.05$, and conclude that $\mu > 175$ mg/dl.

7.4 One-Sample Test for the Mean of a Normal Distribution (Two-Sided Alternatives)

DEFINITION 7.15

A **two-tailed test** is a test in which the values of the parameter being studied (in this case μ) under the alternative hypothesis are allowed to be either *greater than or less than* the values of the parameter under the null hypothesis (μ_0).

Two-Tailed (Sided) Test

Third Test Procedure

$$H_0\colon \mu = \mu_0 \;\; vs. \;\; H_1\colon \mu \neq \mu_0$$

EQUATION 7.10

One-Sample *t* Test for the Mean of a Normal Distribution with Unknown Variance (Two-Sided Alternative)

To test the hypothesis $H_0\colon \mu = \mu_0$ vs. $H_1\colon \mu \neq \mu_0$, with a significance level of α, the best test is based on $t = (\bar{x} - \mu_0)/(s/\sqrt{n})$.

If $|t| > t_{n-1,1-\alpha/2}$
then H_0 is rejected.

If $|t| \le t_{n-1,1-\alpha/2}$
then H_0 is accepted.

Notation: The conditions to use this Test Procedure are:
(1) Normal Distribution.
(2) Population Standard Deviation $\sigma = \sqrt{\sigma^2}$ is Unknown.
(3) The Sample size (n) is small ($n < 30$).

EXAMPLE 7.21

Cardiovascular Disease Test using the one-sample t test at significance level $\alpha = 0.05$ the hypothesis that the mean cholesterol level of recent female Asian immigrants is different from the mean in the general U.S. population, that is:

$$H_0: \mu = 190 \;\; vs \;\; H_1: \mu \neq 190$$

Assuming that cholesterol levels in women ages (21–40) in the United States are approximately normally distributed with mean (μ =190) mg/dL. Blood tests are performed on 28 female Asian immigrants ages (21–40), and the mean level ($\bar{X}$) is 181.52 mg/dL with standard deviation S = 40 mg/dL?

Solution

- Conditions: *We have:*
  1- Normal distribution (Normal population).
  2- The standard deviation σ is unknown (S = 40).
  3- The sample size (n) is small (n = 28 < 30).
- Test Statistic (Calculated Value)

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{181.52 - 190}{40/\sqrt{28}} = -1.122$$

- Critical Value(Tabulated Value)

$$t_{(n-1,\, 1-(\frac{\alpha}{2}))} = t_{(27,\, 0.975)} = 2.052$$

- Decision

We get $t = -1.122$

$|t| = |-1.122| = 1.122$

$|t| = 1.122 < t_{(27,\, 0.975)} = 2.052$

- Conclusion

We accept H_0 at significance level $\alpha = 0.05$, and conclude that $\mu = 190$ mg/dl.

We conclude that the mean cholesterol level of recent Asian immigrants is equal to (*not different*) that of the general U.S. population.

One-Sample Z Test

The test procedure for the one-sample Z test can explained using the three cases of the alternative hypothesis (H_1) as follows:

Test Statistic (Calculated Value)

Case (1)

- Conditions

(1) Normal Distribution.

(2) Population Standard Deviation σ is Known.

(3) Sample size (n) is small ($n < 30$) or is large ($n \geq 30$).

Test Statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

Case (2)

- Conditions

(1) Normal Distribution.

(2) Population Standard Deviation σ is Un Known.

(3) Sample size (n) is large ($n \geq 30$).

Test Statistic

$$Z = \frac{\bar{X} - \mu_0}{S / \sqrt{n}}$$

Notation

If the underlying distribution is unknown or not-normal and the sample size (n) is large ($n \geq 30$) then the central limit theorem can be used as follows:

(a) If population standard deviation σ is known then calculate Z using case (1).

(b) If population standard deviation σ is unknown then calculate Z using case (2).

Rejection Rule

One-Sided Alternative (Lower-Tailed Test)

Hypotheses

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu < \mu_0$$

Rejection Rule

- Reject H_0 at level of significance α if $Z < -Z_{1-\alpha}$
- Accept H_0 at level of significance α if $Z \geq -Z_{1-\alpha}$

where Z = Test Statistic value and $-Z_{1-\alpha}$ = Critical Value from Table 3 in the Appendix.

One-Sided Alternative (Upper-Tailed Test)

Hypotheses

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu > \mu_0$$

Rejection Rule

- Reject H_0 at level of significance α if $Z > Z_{1-\alpha}$
- Accept H_0 at level of significance α if $Z \leq Z_{1-\alpha}$

where Z = Test Statistic value and $Z_{1-\alpha}$ = Critical Value from Table 3 in the Appendix.

Two-Sided Alternative (Two-Tailed Test)

Hypotheses

$$H_0 : \mu = \mu_0 \ \text{ vs } \ H_1 : \mu \neq \mu_0$$

Rejection Rule

- Reject H_0 at level of significance α if $|Z| > Z_{1-\left(\frac{\alpha}{2}\right)}$
- Accept H_0 at level of significance α if $|Z| \leq Z_{1-\left(\frac{\alpha}{2}\right)}$

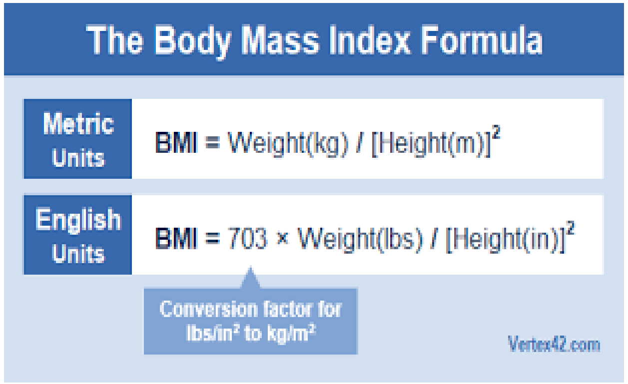
where Z = Test Statistic value and $Z_{1-\left(\frac{\alpha}{2}\right)}$ = Critical Value from Table 3 in the Appendix.

Example

The body mass index (BMI) of a random sample of size 14 healthy adult males has a mean of 30.5. Can we conclude at α = 0.05 (or 5%) that the mean BMI of the population (μ) is lower than 36 assuming that the population is normally distributed with a standard deviation of 10.6392?

Solution

$$H_0 : \mu = 36 \ \text{ vs } \ H_1 : \mu < 36$$



- Conditions

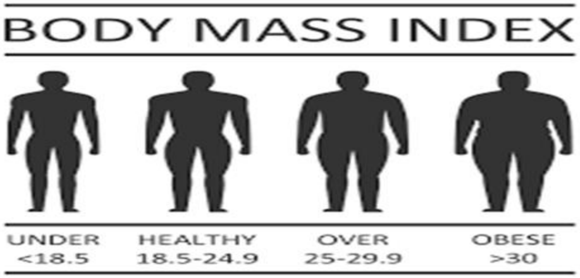
(1) Normal Distribution.

(2) Population Standard Deviation σ is Known.

(3) Sample size (n) is small ($n < 30$) (*not important*).

- **Test Statistic Value**

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{30.5 - 36}{10.6392 / \sqrt{14}} = -1.934$$



- **Rejection Rule (One-Sided Lower Tailed Test)**
  - Reject H_0 at level of significance α if $Z < -Z_{1-\alpha}$
  - Accept H_0 at level of significance α if $Z \geq -Z_{1-\alpha}$
- **Critical Value**

$$-Z_{1-\alpha} = -Z_{1-0.05} = -Z_{0.95} = -1.645$$ → Refer to Table 3 in the Appendix

- **Decision**

We get

$$Z = -1.934 < -Z_{0.95} = -1.645$$

Then Reject $H_0: \mu = 36$ and therefore Accept $H_1: \mu < 36$ at level of significance α= 0.05 , and conclude that mean BMI of the population (μ) is lower than 36.

Exercise

The body mass index (BMI) of a random sample of size 34 healthy adult males has a mean of 38.5. Can we conclude at α = 0.05 (or 5%) that the mean BMI of the population (μ) is higher than 36 assuming that the population is normally distributed with a standard deviation of 10.6392?

Answer: No because we will Accept $H_0: \mu = 36$ and Reject $H_1: \mu > 36$ at α = 0.05.

Exercise

The body mass index (BMI) of a random sample of size 41 healthy adult males has a mean of 30.5 and a standard deviation of 10.6392. Can we conclude at α = 0.05 (or 5%) that the mean BMI of the population (μ) is different from 36 assuming that the population is normally distributed with?

Answer

- Conditions

(1) Normal Distribution.
(2) Population Standard Deviation σ is Un Known.
(3) Sample size (n) is large ($n \geq 30$).

- Test Statistic

$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = -3.310$$

- Decision

We get

$$|Z| = |-3.310| = 3.310 > Z_{1-\left(\frac{\alpha}{2}\right)} = Z_{0.975} = 1.96$$

Then Reject $H_0: \mu = 36$ and therefore Accept $H_1: \mu \neq 36$ at level of significance α= 0.05 , and conclude that mean BMI of the population (μ) is different from 36.

Exercise

Study **Example 7.25** (Cardiovascular Disease) page 228 in the textbook.

7.7 Sample-Size Determination

For planning purposes, we frequently need some idea of an appropriate sample size (n) for investigation before a study actually begins. In this section, we will learn how to determine the sample size (n) by using the a method based on CI width (CI length) given as follows:

EQUATION 7.24

Sample-Size Estimation Based on CI Width

Suppose we wish to estimate the mean of a normal distribution with sample variance s^2 and require that the two-sided $100\% \times (1-\alpha)$ CI for μ be no wider than L. The number of subjects needed is approximately

$$n = 4z_{1-\alpha/2}^2\, s^2 / L^2$$

Notation: The value of the sample size (n) obtained by using Equation 7.24 should be rounded up to the next integer.

EXAMPLE 7.45

Cardiology Find the value for the minimum sample size (n) needed to estimate the change in heart rate (μ) if we require that the two-sided 95% CI for μ be no wider than 5 beats per minute and the sample standard deviation for change in heart rate equals 10 beats per minute?

Solution

Step(1)

$(1-\alpha)\,100\% = 95\%$

$1 - \alpha = 0.95$

$\alpha = 0.05$

$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$

$1 - \frac{\alpha}{2} = 1 - 0.025 = 0.975$

$Z_{1-\frac{\alpha}{2}} = Z_{0.975} = 1.96$

Refer to Table 3 in the Appendix

Step(2)

$$n = \frac{4\left(Z_{1-(\alpha/2)}\right)^2 s^2}{L^2}$$

$$n = \frac{4\,(1.96)^2\,10^2}{5^2}$$

$$n = \frac{(15.3664)(100)}{25}$$

$$n = 61.4656$$

$$n \cong 62$$

Exercise

Cardiology Find the value for the minimum sample size (n) needed to estimate the change in heart rate (μ) if we require that the two-sided 90% CI for μ be no wider than 3 beats per minute and the sample standard deviation for change in heart rate equals 10 beats per minute?

Answer:

$Z_{1-\frac{\alpha}{2}} = Z_{0.95} = 1.645$

$n = 120.268$ implies that $n \cong 121$

7.9 One-Sample Inference for the Binomial Distribution

Normal-Theory Methods

In this section, we will study the one-sample inference for the population proportion of a binomial distribution (p) based on the sample proportion of cases $\hat{p}$ assuming the normal approximation to the binomial distribution is valid. We know that under $H_0: p = p_0$ and when $np_0q_0 \geq 5$ where $q_0 = 1 - p_0$, then the sampling distribution of the sample proportion ($\hat{p}$) will be normal distribution as follows:

$$\hat{p} \sim N(p_0, \frac{p_0 q_0}{n})$$

In the rest of this section, we focus primarily on two-sided tests because they are much more widely used in the literature. Now, to test at level of significance (α) the following two-sided alternative hypotheses: $H_0 : p = p_0$ *vs* $H_1 : p \neq p_0$. A continuity-corrected version of the test statistic Z can be used. Thus, the test takes the following form:

EQUATION 7.27

One-Sample Test for a Binomial Proportion—Normal-Theory Method (Two-Sided Alternative)

Let the test statistic $z_{corr} = \left(\left| \hat{p} - p_0 \right| - \frac{1}{2n} \right) \Big/ \sqrt{p_0 q_0 / n}$.

If $z_{corr} > z_{1-\alpha/2}$, then H_0 is rejected. If $z_{corr} < z_{1-\alpha/2}$, then H_0 is accepted. This test should only be used if $np_0q_0 \geq 5$.

EXAMPLE 7.49

Cancer Consider a breast-cancer problem were we interested in the effect of having a family history of breast cancer on the incidence of breast cancer. Suppose that 400 of the 10,000 (*that is the value of* $\hat{p} = 400/10000 = 0.04$) women ages (50–54) sampled whose mothers had breast cancer had breast cancer themselves at some time in their lives. Given large studies, assume the prevalence rate of breast cancer for U.S. women in this age group is about 2% $(p = p_0 = 0.02)$. If p = prevalence rate of breast cancer in (50-to-54) year-old women whose mothers have had breast cancer, then we want to test the two-sided alternative hypothesis:

$$\mathrm{H}_0: p = 0.02 \quad vs \quad \mathrm{H}_1: p \neq 0.02$$

at level of significance $\alpha = 0.05$?

Solution

- Test Statistic Value

$$Z_{corr} = \frac{|\hat{p} - p_0| - \frac{1}{2n}}{\sqrt{p_0 q_0 / n}}$$

$$= \frac{|.04 - .02| - \frac{1}{2(10,000)}}{\sqrt{.02(.98)/10,000}} = \frac{.0200}{.0014} = 14.3$$

Table 3 Appendix

- Critical Value

$$Z_{1-\frac{\alpha}{2}} = Z_{0.975} = 1.96$$

- Decision

We get

$$Z_{corr} = 14.3 > Z_{1-\frac{\alpha}{2}} = Z_{0.975} = 1.96$$

It follows that $\mathrm{H}_0: p = 0.02$ can be rejected using a two-sided test with α = 0.05 and conclude that $p \neq 0.02$.

p – value

DEFINITION 7.13

The ***p*-value** for any hypothesis test is the α level at which we would be indifferent between accepting or rejecting H_0 given the sample data at hand. That is, the *p*-value is the α level at which the given value of the test statistic (such as *t*) is on the borderline between the acceptance and rejection regions.

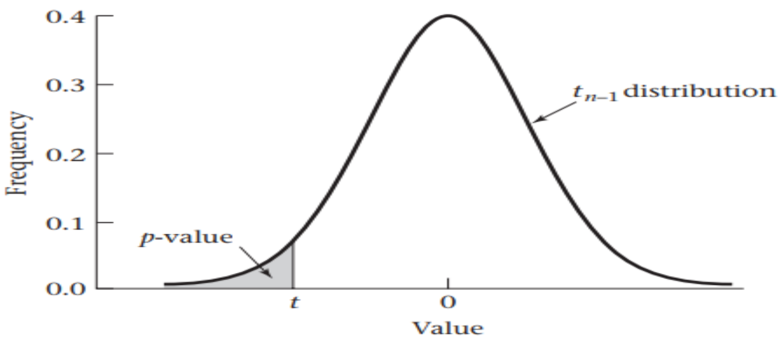
Notation: p is the area to the left of t under a $t_{(n-1)}$ distribution as given in the following equation:

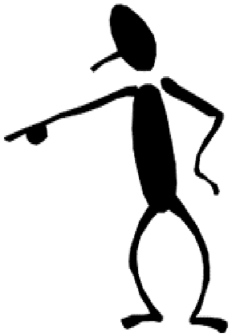
EQUATION 7.3

$$p = Pr(t_{n-1} \leq t)$$

The p-value can be displayed as shown in Figure 7.1.

FIGURE 7.1 Graphic display of a *p*-value





EXAMPLE 7.12

Suppose that we use the one-sample t test to test the hypothesis:

Ho: μ = 120   vs. H1: μ < 120

based on the birthweight data for $n = 100$ and $\alpha = 0.05$, and after we compute the value of the test statistic we get $t =$ -2.08. Compute the p-value for the birthweight data?

Solution

From Equation 7.3, the p-value can be calculated as follows:

$$p = P\left(t_{(n-1)} \leq t\right) = P(t_{99} \leq -2.08) = 0.020$$

which is the p-value.

An alternative definition of a p-value that is useful in other hypothesis-testing problems is as follows:

DEFINITION 7.14

The ***p*-value** can also be thought of as the probability of obtaining a test statistic as extreme as or more extreme than the actual test statistic obtained, given that the null hypothesis is true.

p-value Method (Approach)

The p-value Method (Approach) can be used to establish whether results from hypothesis tests are statistically significant:

(1) Calculate the exact p-value.

(2) If p-value < α, then H_0 is rejected (*results are statistically significant*).

(3) If p-value ≥ α, then H_0 is accepted (*results are not statistically significant*).

Example

Suppose that we use the one-sample t test to test the hypothesis:

Ho: μ = 120 vs. H1: μ < 120

based on the birthweight data for $n = 100$ and $\alpha = 0.05$, and after we compute the value of the test statistic we get $t =$ -2.08 and the p-value for the birthweight data is as follows:

$$p = P\left(t_{(n-1)} \leq t\right) = P(t_{99} \leq -2.08) = 0.020$$

Assess the statistical significance of the birthweight data?

Solution

Because the p-value is $p = 0.020 < \alpha = 0.05$, then H_0 is rejected and the results would be considered statistically significant and we would conclude that the true mean birthweight (μ) is significantly lower in this hospital than in the national average in the general population.

Example

Suppose that we use the one-sample t test to test the hypothesis:

Ho: μ = 120 vs. H1: μ < 120

based on the birthweight data for $n = 10$ and $\alpha = 0.05$, and after we compute the value of the test statistic we get $t =$ -1.32 and the p-value for the birthweight data is as follows:

$$p = P\left(t_{(n-1)} \leq t\right) = P(t_9 \leq -1.32) = 0.110$$

Assess the statistical significance of the birthweight data?

Solution

Because the p-value is $p = 0.110 > \alpha = 0.05$, then H_0 is accepted and the results would be considered not statistically significant and we would conclude that the true mean birthweight (μ) does not differ (equal) significantly in this hospital to the national average in the general population.

How to Calculate the p-value for the One-Sample t-Test of the Mean (μ):

(1) For a one-sided lower-tailed t–test:

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu < \mu_0$$

the p-value can be calculated as follows:

$$p = P\left(t_{(n-1)} \leq t\right)$$

(2) For a one-sided upper-tailed t–test:

$$H_0 : \mu = \mu_0 \ \text{ vs } \ H_1 : \mu > \mu_0$$

the p-value can be calculated as follows:

$$p = P\left(t_{(n-1)} > t\right) = 1 - P\left(t_{(n-1)} \le t\right)$$

(3) For a two-sided (two-tailed) t–test:

$$H_0 : \mu = \mu_0 \ \text{ vs } \ H_1 : \mu \neq \mu_0$$

the p-value is computed in two different ways, depending on whether t is less than or greater than 0 as follows:

$$p = \begin{cases} 2 \times P\left(t_{(n-1)} \le t\right) & , \quad if\ t \le 0 \\ 2 \times \left[1 - P\left(t_{(n-1)} \le t\right)\right] & , \quad if\ t > 0 \end{cases}$$

How to Calculate the p-value for the One-Sample Z-Test of the Mean (μ):

(1) For a one-sided lower-tailed Z–test:

$$H_0 : \mu = \mu_0 \ \text{ vs } \ H_1 : \mu < \mu_0$$

the p-value can be calculated as follows:

$$p = P(Z \le z) = \Phi(z)$$

(2) For a one-sided upper-tailed Z–test:

$$H_0 : \mu = \mu_0 \ \text{ vs } \ H_1 : \mu > \mu_0$$

the p-value can be calculated as follows:

$$p = P(Z > z) = 1 - P(Z \le z) = 1 - \Phi(z)$$

(3) For a two-sided (two-tailed) Z–test:

$$H_0 : \mu = \mu_0 \ \text{ vs } \ H_1 : \mu \ne \mu_0$$

the p-value is computed in two different ways, depending on whether z is less than or greater than 0 as follows:

$$p = \begin{cases} 2 \times P(Z \le z) = 2 \times \Phi(z) & , \quad if\ z \le 0 \\ 2 \times [1 - P(Z \le z)] = 2 \times [1 - \Phi(z)] & , \quad if\ z > 0 \end{cases}$$

EXAMPLE 7.25

Cardiovascular Disease Consider a cholesterol data. Assume that the standard deviation is known to be 40 and the sample size is 200. Assess the significance of the results if we want to test:

$$H_0 : \mu = 190 \ \text{ vs } \ H_1 : \mu \ne 190$$

at $\alpha = 0.05$? Where the value of the sample mean is $\bar{X}$ = 181.52?

Solution

- The test statistic is: $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} = \frac{181.52-190}{40/\sqrt{200}}$ = -2.998

- The p-value is:

  We have Z = -2.998 < 0 then the p-value can be calculated as follows:

$$p = 2 \times P(Z \leq z) = 2 \times \Phi(z) = 2 \times \Phi(-2.998)$$
$$= 2 \times [1- \Phi(2.998)]$$
$$= 2 \times [1- \Phi(3.00)]$$
$$= 2 \times [1- 0.9987]$$
$$= 2 \times 0.0013$$
$$= 0.0026 \approx 0.003$$

- Hypothesis Test using the Critical Value Method:

  We get $|Z| = |-2.998| = 2.998 > Z_{1-\left(\frac{\alpha}{2}\right)} = Z_{0.975} = 1.96$

  Then Reject H_0: μ = 190 and Accept H_1: μ ≠ 190 at α= 0.05, and conclude that the mean of the population (μ) is different from 190.

- Hypothesis Test using the p-Value Method:

  We get p-value = $0.003 < \alpha = 0.05$ then Reject H_0: μ = 190 and Accept H_1: μ ≠ 190 at α= 0.05, and conclude that the population (μ) is significantly different from 190.

How to Calculate the p-value for the One-Sample Z-Test of the Proportion (p):

(3) For a two-sided (two-tailed) Z–test:

$$H_0 : p = p_0 \text{ vs } H_1 : p \neq p_0$$

the p-value is computed in two different ways, depending on whether z is less than or greater than 0 as follows:

$$p = 2 \times [1 - \Phi(Z_{corr})]$$

(Twice the area to the right of Z_{corr} under an N(0, 1) curve)

EXAMPLE 7.49

Given that: $H_0: p = 0.02$ *vs* $H_1: p \neq 0.02$

Assess the statistical significance of the data at level of significance $\alpha = 0.05$?

Solution

- The test statistic is:

$$Z_{corr} = \frac{|\hat{p} - p_0| - \frac{1}{2n}}{\sqrt{p_0 q_0 / n}}$$

$$= \frac{|.04 - .02| - \frac{1}{2(10{,}000)}}{\sqrt{.02(.98)/10{,}000}} = \frac{.0200}{.0014} = 14.3$$

We have $Z_{corr} = 14.3$ then the p-value is:

$$p = 2 \times [1 - \Phi(Z_{corr})]$$
$$= 2 \times [1 - \Phi(14.3)]$$
$$= 2 \times [1 - 1]$$
$$= 2 \times [0]$$
$$= 0 < 0.05$$

Thus, Reject H0, the results are very highly significant. Then $p \neq 0.02$.

Problems: 7.1 – 7.8, 7.12 – 7.14, 7.23– 7.25.