Estimation 6

Solved Problems-Number (3)

Interval Estimation – Confidence Interval for the Population Mean (μ)

Although the point estimator ($\hat{\theta}$) is a good estimator for the population parameter (θ), for example the sample mean ($\overline{X}$) is a good point estimator for the population mean (μ), it is more meaningful to estimate θ by an interval that communicates information regarding the probable magnitude of θ, this interval is known as the confidence interval.

Definition: Interval Estimation

An interval estimator of a population parameter θ is an interval of the form:

$$\hat{\theta}_L < \theta < \hat{\theta}_U$$

where $\hat{\theta}_L$ and $\hat{\theta}_U$ depends on the value of the statistic $\hat{\theta}$ for a particular random sample $X_1, X_2, \dots, X_n$ and also on the sampling distribution of $\hat{\theta}$.

Definition: Confidence Interval

The interval $\hat{\theta}_L < \theta < \hat{\theta}_U$ computed from the selected random sample, is then called a $(1-\alpha)100\%$ confidence interval, the fraction $(1-\alpha)$ is called confidence coefficient (confidence level) or the degree of confidence and the end points $\hat{\theta}_L$ and $\hat{\theta}_U$ are called the lower and upper confidence limits. The general formula for constructing a confidence interval (CI) is given as follows:

CI = Point Estimator ± [(Critical Value)(Standard Error)]

where

- Point Estimator: is the sample statistic estimating the population parameter of interest.
- Critical Value: is a table value based on the sampling distribution of the point estimator and the desired confidence level.
- Standard Error: is the standard deviation of the point estimator.

$$P(\hat{\theta}_L < \theta < \hat{\theta}_U) = 1-\alpha \quad \text{for } 0 < \alpha < 1$$

In this section, we will learn how to:

Construct and interpret the (1 – α)100% confidence interval for the population mean (μ).

- In the case that the underlying distribution is Normal Distribution, three cases for the Confidence Interval of the Population Mean (μ) are discussed:

  (1) when the Population Standard Deviation σ is Known (One Case).
  (2) when the Population Standard Deviation σ is Unknown (Two Cases).

- Also, we will study a fourth case for the Confidence Interval of the population mean (μ) known as the Central Limit Theorem (CLT) Case.

Case (1): Confidence Interval for the Population Mean (μ) of a Normal Distribution (σ is Known)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n taken from a normal distribution, $N(\mu, \sigma^2)$. If the population variance σ^2 (or population standard deviation $\sigma = \sqrt{\sigma^2}$) is known, then for ($n < 30$ (small) or $n \geq 30$ (large)), the (1 – α) × 100% confidence interval (CI) for the population mean μ of a normal distribution can be constructed as follows:

$$\text{CI} = \left(\bar{X} - Z_{1-\left(\frac{\alpha}{2}\right)} \times \frac{\sigma}{\sqrt{n}} \quad , \quad \bar{X} + Z_{1-\left(\frac{\alpha}{2}\right)} \times \frac{\sigma}{\sqrt{n}}\right)$$

Notation: The conditions to use this confidence interval (CI) are:

(1) Normal Distribution.

(2) Population Standard Deviation $\sigma = \sqrt{\sigma^2}$ is Known ($n < 30$ or $n \geq 30$).

Normal Critical Values for Confidence Levels

Confidence Level, (1 – α)	Critical Value, $Z_{1-\left(\frac{\alpha}{2}\right)}$
99%	2.575
95%	1.96
90%	1.645

Question (1)

Suppose it is known that in a certain large human population cranial length is normally distributed with a mean of μ mm and standard deviation is 12.7 mm. A random sample of size 10 is taken from this population showed that the sample mean ($\overline{X}$) is 190mm. Construct the 99% confidence interval (CI) for the population mean of cranial lengths (μ)?

Solution

(i) We have the following:

(1) X = cranial length $\sim N(\mu, (12.7)^2)$.

(2) The population standard deviation $\sigma = \sqrt{\sigma^2}$ is known ($\sigma = 12.7$ mm).

(3) The sample size $n = 10 < 30$ is small (not important).

$$n = 10\ ;\ \overline{X} = 190 \text{ mm}\ ;\ \sigma = 12.7 \text{ mm}\ ;\ (1-\alpha)\ 100\ \% = 99\ \%$$

(ii) The 99% confidence interval (CI) for the population mean of cranial lengths (μ) can be calculated as follows:

Step(1)

$(1-\alpha)$ 100 % = 99% implies that $Z_{1-\left(\frac{\alpha}{2}\right)} = 2.575$

Step(2)

The formula for the $(1-\alpha)100\%$ confidence interval (CI) for the population mean (μ) in this case is given as follows:

$$\text{CI} = \overline{X} \pm Z_{1-\left(\frac{\alpha}{2}\right)} \frac{\sigma}{\sqrt{n}}$$

$$\text{Lower Limit} = 190 - \left[(2.575)\left(\frac{12.7}{\sqrt{10}}\right)\right] = 190 - 10.34 = 179.66\ mm$$

$$\text{Upper Limit} = 190 + \left[(2.575)\left(\frac{12.7}{\sqrt{10}}\right)\right] = 190 + 10.34 = 200.34\ mm$$

Then the 99% confidence interval for the population mean of cranial lengths (μ) is CI = (L, U) = (179.66, 200.34) mm.

Conclusion

With 99% confidence interval we can say that the mean of cranial lengths (μ) of all human in this population is between 179.66 mm and 200.34 mm.

Case (2): Confidence Interval for the Population Mean (μ) of a Normal Distribution
(σ is Unknown and n is Large)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n taken from a normal distribution, $N(\mu, \sigma^2)$. If the population variance σ^2 (or population standard deviation $\sigma = \sqrt{\sigma^2}$) is unknown and sample size n is large ($n \geq 30$), then replace $\sigma = \sqrt{\sigma^2}$ by the sample standard deviation $S = \sqrt{S^2}$ and the $(1 - \alpha) \times 100\%$ confidence interval (CI) for the population mean μ of a normal distribution can be constructed as follows:

$$CI = \left(\overline{X} - Z_{1-\left(\frac{\alpha}{2}\right)} \times \frac{S}{\sqrt{n}} \quad , \quad \overline{X} + Z_{1-\left(\frac{\alpha}{2}\right)} \times \frac{S}{\sqrt{n}}\right)$$

Notation: The conditions to use this confidence interval (CI) are:

(1) Normal Distribution.

(2) Population Standard Deviation $\sigma = \sqrt{\sigma^2}$ is Unknown.

(3) The sample size (n) is large ($n \geq 30$).

Question (2)

Suppose it is known that in a certain large human population cranial length is normally distributed with a mean of μ mm. A random sample of size 100 is taken from this population showed that the sample mean ($\overline{X}$) is 190 mm and the sample standard deviation is 12.7 mm. Construct the 99% confidence interval (CI) for the population mean of cranial lengths (μ)?

Solution

(i) We have the following:

(1) X = cranial length$\sim N(\mu, \sigma^2)$.

(2) The population standard deviation $\sigma = \sqrt{\sigma^2}$ is unknown ($S = 12.7$ mm).

(3) The sample size $n = 100 > 30$ is large (important).

$n = 100$; $\overline{X}$ = 190 mm ; S = 12.7 mm ; $(1 - \alpha)$ 100 % = 99 %

(ii) The 99% confidence interval (CI) for the population mean of cranial lengths (μ) can be calculated as follows:

Step(1)

$(1 - \alpha)$ 100 % = 99% implies that $Z_{1-\left(\frac{\alpha}{2}\right)}$ = 2.575

Step(2)

The formula for the $(1-\alpha)100\%$ confidence interval (CI) for the population mean (µ) in this case is given as follows:

$$\text{CI} = \overline{X} \pm Z_{1-(\frac{\alpha}{2})} \frac{S}{\sqrt{n}}$$

Lower Limit = $190 - \left[(2.575)\left(\frac{12.7}{\sqrt{100}}\right)\right] = 190 - 3.27025 = 186.73\ mm$

Upper Limit = $190 + \left[(2.575)\left(\frac{12.7}{\sqrt{100}}\right)\right] = 190 + 3.27025 = 193.27\ mm$

Then the 99% confidence interval for the population mean of cranial lengths (µ) is CI = (L, U) = (186.73, 193.27) mm.

Conclusion

With 99% confidence interval we can say that the mean of cranial lengths (µ) of all human in this population is between 186.73 mm and 193.27 mm.

Case (3): Confidence Interval for the Population Mean () of a Normal Distribution (σ is Unknown and n is Small)

Let $X_1, X_2, \dots, X_n$ be a random sample of size (n) taken from a normal distribution, $N(\mu, \sigma^2)$. If the population variance σ^2 (or the population standard deviation $\sigma = \sqrt{\sigma^2}$) is unknown and sample size n is small (n < 30), then replace $\sigma = \sqrt{\sigma^2}$ by the sample standard deviation $S = \sqrt{S^2}$ and the $(1-\alpha) \times 100\%$ confidence interval (CI) for the population mean (μ) of a normal distribution will be constructed as follows:

$$\text{CI} = \left(\overline{\text{X}} - t_{\left(n-1\,,\,1-\left(\frac{\alpha}{2}\right)\right)} \times \frac{\text{S}}{\sqrt{\text{n}}}\,,\overline{\text{X}} + t_{\left(n-1\,,\,1-\left(\frac{\alpha}{2}\right)\right)} \times \frac{\text{S}}{\sqrt{\text{n}}}\right)$$

Notation: The conditions to use this confidence interval (CI) are:

(1) Normal Distribution.

(2) Population Standard Deviation $\sigma = \sqrt{\sigma^2}$ is Unknown.

(3) The Sample size (n) is small (n < 30).

Question (3)

Suppose it is known that in a certain large human population cranial length is normally distributed with a mean of μ mm. A random sample of size 10 is taken from this population showed that the sample mean ($\bar{X}$) is 190 mm and the sample standard deviation is 12.7 mm. Construct the 99% confidence interval (CI) for the population mean of cranial lengths (μ)?

Solution

We have:

(1) X = cranial length $\sim N(\mu, \sigma^2)$.

(2) The population standard deviation $\sigma = \sqrt{\sigma^2}$ is unknown ($S = 12.7$ mm).

(3) The sample size $n = 10 < 30$ is small (important).

$$n = 10 \; ; \; \bar{X} = 190 \text{ mm} \; ; \; S = 12.7 \text{ mm} \; ; \; (1 - \alpha)\, 100\,\% = 99\,\%$$

Step(1)

$(1 - \alpha)$ 100 % = 99 %

$1 - \alpha = 0.99$, $\alpha = 0.01$, $\frac{\alpha}{2} = \frac{0.01}{2} = 0.005$, $1 - \frac{\alpha}{2} = 1 - 0.005 = 0.995$

$d = n - 1 = 10 - 1 = 9$ implies that $t_{(n-1,\; 1-\frac{\alpha}{2})} = t_{(9,\; 0.995)} = 3.250$

Step(2)

The formula for the $(1 - \alpha)$100% confidence interval (CI) for the population mean (μ) in this case is given as follows:

$$CI = \bar{X} \pm t_{(n-1,\; 1-\frac{\alpha}{2})} \frac{S}{\sqrt{n}}$$

Lower Limit = $190 - \left[(3.250)\left(\frac{12.7}{\sqrt{10}}\right)\right] = 190 - 13.052 = 176.95\; mm$

Upper Limit = $190 + \left[(3.250)\left(\frac{12.7}{\sqrt{10}}\right)\right] = 190 + 13.052 = 203.52\; mm$

Then the 99% confidence interval for the population mean of cranial lengths (μ) is CI = (L, U) = (176.95, 203.52) mm.

Conclusion

With 99% confidence interval we can say that the mean of cranial lengths (μ) of all human in this population is between 176.95 mm and 203.52 mm.

Central Limit Theorem (CLT) Case for Confidence Interval

To use the central limit theorem (CLT) to construct the (1-α)100% confidence interval for the population mean (μ) the following two conditions should be satisfied:

1- Unknown distribution (population) or non-normal distribution with a population mean (μ) and a population variance (σ^2) or a population standard deviation ($\sigma = \sqrt{\sigma^2}$). If σ is unknown replace it by the sample standard deviation (S).

2- The sample size (n) is large ($n \geq 30$).

Then we assume the sampling distribution of the sample mean ($\overline{X}$) to be approximately normally distributed by using the central limit theorem (CLT) and therefore the (1-α)100% confidence interval for the population mean (μ) can be constructed using the following formula:

$$CI = \left(\overline{X} - Z_{1-\left(\frac{\alpha}{2}\right)} \times \frac{\sigma}{\sqrt{n}} \quad , \quad \overline{X} + Z_{1-\left(\frac{\alpha}{2}\right)} \times \frac{\sigma}{\sqrt{n}}\right)$$

Question (4)

Suppose that we wish to estimate the mean number of heart beats per minute for a certain population (μ). The average number of heart beats per minute for a random sample of size 49 subjects was found to be 90. Previous research has shown that the standard deviation for the population to be about 10. Find the 90% confidence interval (CI) for the population mean (μ)?

Solution

Now, since the above conditions are satisfied, we draw on the central limit theorem and assume that the sampling distribution of the sample mean ($\overline{X}$) to be approximately normally distributed.

The (1 - α) × 100 % = 90% confidence interval for the population mean (μ) of the number of heart beats per minute can be constructed as follows:

Step(1)

$(1-\alpha)$ 100 % = 90% implies that $Z_{1-\left(\frac{\alpha}{2}\right)} = 1.645$

$$n = 49 \,;\ \overline{X} = 90 \,;\ \sigma = 10 \,;\ (1-\alpha)\ 100\ \% = 90\ \%$$

Step(2)

The $(1-\alpha)100\%$ = 90% confidence interval (CI) for the population mean (μ) of the number of heart beats per minute can be calculated as follows:

$$\text{CI} = \bar{X} \pm Z_{1-(\frac{\alpha}{2})} \frac{\sigma}{\sqrt{n}} = 90 \pm \left[(1.645)\left(\frac{10}{\sqrt{49}}\right)\right] = 90 \pm 2.35 = (87.65\ ,\ 92.35)$$

Conclusion

With 90% confidence interval we can say that the mean number of heart beats per minute for all subjects in the population (μ) is between 87.65 and 92.35.

Exercises

Exercise (1)

In a length of hospitalization study conducted by several cooperating hospitals from Jordan, a random sample of size 64 peptic ulcer patients was drawn from a list of all peptic ulcer patients ever admitted to the participating hospitals and the length of hospitalization per admission was determined for each. The mean length of hospitalization was found to be 8.25 days. Find the 99% confidence interval for the mean length of hospitalization per admission for all peptic ulcer patients ever admitted to the participating hospitals if the population standard deviation is known to be 3 days?

Answer: CI = (7.28, 9.22).

Exercise (2)

Suppose average pizza delivery times are normally distributed with an unknown population mean and a population standard deviation of 6 minutes. A random sample of size 28 pizza delivery restaurants is taken and has a sample mean delivery time of 36 minutes. Find a 90% confidence interval for the population mean delivery time (μ)?

Answer: CI = (34.135, 37.865).

Exercise (3)

A random sample of 120 students from a large university yields mean GPA 2.71 with sample standard deviation 0.51. Construct a 90% confidence interval for the mean GPA of all students at the university if the GPA of all students at the university is normally distributed?

Answer: CI = (2.63, 2.79).

Exercise (4)

The contents of 7 similar containers of sulphuric acid are 9.8, 10.2, 10.4, 9.8, 10, 10.2, 9.6 litters. Find a 95% confidence interval for the mean of all such containers assuming a normal distribution?

Answer: CI = (9.738, 10.262).

Exercise (5)

Suppose scores on Biostatistics exams are normally distributed with an unknown population mean and a population standard deviation of 3 points. A random sample of size 36 scores is taken and gives a sample mean (sample mean score) of 68. Find a 90% confidence interval for the population mean exam score (the mean score on all exams)?

Answer: CI = (67.1775, 68.8225).
