Hypothesis Testing: One-Sample Inference

7

Solved Problems-Number (1)

One-Sample Inference

Hypothesis Testing for the Population Mean (μ)

7.1- Introduction

In chapter 6 we discuss the methods of point and interval estimation for population mean (μ) and population proportion (p) parameters of various distributions. In this chapter (chapter 7), some of the basic concepts of hypothesis testing are developed and applied to one-sample problems of statistical inference. In a one-sample problem, hypotheses are specified about a single distribution; in a two-sample problem, two different distributions are compared.

7.2- General Concepts

DEFINITION 7.1 The **null hypothesis,** denoted by H_0, is the hypothesis that is to be tested. The alternative hypothesis, denoted by H_1, is the hypothesis that in some sense contradicts the null hypothesis.

Notation

We will assume the underlying distribution is normal under either hypothesis.

An example for how the two hypotheses H_0 and H_1 can be written isgiven as in the following form:

EQUATION 7.1 $H_0: \mu = \mu_0$ vs. (versus) $H_1: \mu < \mu_0$

Types of Errors in Statistical Hypotheses Testing

DEFINITION 7.2 The probability of a **type I error** is the probability of rejecting the null hypothesis when H_0 is true.

Example

Probability of deciding that the mean birthweight in the hospital was lower than 120 oz ($\mu < 120$) when in fact it was 120 oz ($\mu = 120$). → **Type I Error**

DEFINITION 7.3 The probability of a **type II error** is the probability of accepting the null hypothesis when H_1 is true. This probability is a function of μ as well as other factors.

Example

Probability of deciding that the mean birthweight in the hospital was lower than 120 oz ($\mu = 120$) when in fact it was 120 oz ($\mu < 120$). → **Type II Error**

Notation: The Greek letters α and β represent the probabilities of Type I and Type II Errors.

DEFINITION 7.4 The probability of a **type I error** is usually denoted by α and is commonly referred to as the **significance level** of a test.

α = Probability of a Type I Error
= **P(Type I Error)**
= P(Rejecting H_0 when H_0 is true)
→ Rejecting a Good H_0

DEFINITION 7.5 The probability of a **type II error** is usually denoted by β.

β = Probability of a Type II Error
= **P(Type II Error)**
= P(Not Rejecting H_0 when H_0 is false)
→ Accepting a Bad H_0

DEFINITION 7.6 The power of a test is defined as

$$1 - \beta = 1 - \text{probability of a type II error} = Pr(\text{rejecting } H_0 | H_1 \text{ true})$$

Notation: α and β should be as small as possible because they are probabilities of errors.

Notation: Our general strategy in hypothesis testing is to fix α at some specific level (for example, 0.10, 0.05, 0.01, . . .) and to use the test that minimizes β or, equivalently, maximizes the power (1- β).

Types of Regions in Statistical Hypotheses Testing

DEFINITION 7.7 The **acceptance region** is the range of values of $\bar{x}$ for which H_0 is accepted.

DEFINITION 7.8 The **rejection region** is the range of values of $\bar{x}$ for which H_0 is rejected.

Types of Tests in Statistical Hypotheses Testing

DEFINITION 7.9 A **one-tailed test** is a test in which the values of the parameter being studied (in this case μ) under the alternative hypothesis are allowed to be either greater than or less than the values of the parameter under the null hypothesis (μ_0), *but not both*.

One-Tailed (Sided) Test Types

$H_0: \mu = \mu_0 \;\; vs. \;\; H_1: \mu < \mu_0$ [lower-tailed (left-tailed) test]
$H_0: \mu = \mu_0 \;\; vs. \;\; H_1: \mu > \mu_0$ [upper-tailed (right-tailed) test]

DEFINITION 7.15 A **two-tailed test** is a test in which the values of the parameter being studied (in this case μ) under the alternative hypothesis are allowed to be either *greater than or less than* the values of the parameter under the null hypothesis (μ_0).

Two-Tailed (Sided) Test

$H_0: \mu = \mu_0 \;\; vs. \;\; H_1: \mu \neq \mu_0$

DEFINITION 7.12 The general approach in which we compute a test statistic and determine the outcome of a test by comparing the test statistic with a critical value determined by the type I error is called the **critical-value method** of hypothesis testing.

7.3 One-Sample t-Test for the Mean of a Normal Distribution

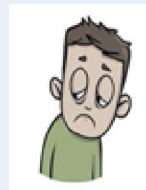
Notation: The conditions to use this **Test Procedure** are:
(1) Normal Distribution.
(2) Population Standard Deviation $\sigma = \sqrt{\sigma^2}$ is Unknown.
(3) The Sample size (n) is small ($n < 30$).

(One-Sided Alternatives – Lower Tailed Test)

EQUATION 7.2

One-Sample *t* Test for the Mean of a Normal Distribution with Unknown Variance (Alternative Mean < Null Mean)

To test the hypothesis $H_0: \mu = \mu_0, \sigma$ unknown vs. $H_1: \mu < \mu_0, \sigma$ unknown with a significance level of α, we compute



$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$ → Test Statistic

If $t < -t_{(n-1, 1-\alpha)}$ then we reject H_0.
If $t \geq -t_{(n-1, 1-\alpha)}$ then we do not reject (accept) H_0.

$-t_{(n-1, 1-\alpha)}$ → Critical Value

(One-Sided Alternatives – Upper Tailed Test)

EQUATION 7.6

One-Sample *t* Test for the Mean of a Normal Distribution with Unknown Variance (Alternative Mean > Null Mean)

To test the hypothesis

$$H_0: \mu = \mu_0 \quad \text{vs.} \quad H_1: \mu > \mu_0$$

with a significance level of α, the best test is based on t, where

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

If $t > t_{(n-1, 1-\alpha)}$ then we reject H_0.
If $t \leq t_{(n-1, 1-\alpha)}$ then we do not reject (accept) H_0.

(Two-Sided Alternatives-TwoTailed)

EQUATION 7.10

One-Sample t Test for the Mean of a Normal Distribution with Unknown Variance (Two-Sided Alternative)

To test the hypothesis $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$, with a significance level of α, the best test is based on $t = (\bar{x} - \mu_0)/(s/\sqrt{n})$.

If $|t| > t_{n-1,1-\alpha/2}$
then H_0 is rejected.

If $|t| \leq t_{n-1,1-\alpha/2}$
then H_0 is accepted.

Question (1)

Suppose it is known that in a certain large human population cranial length is normally distributed. A random sample of size 10 is taken from this population showed that the sample mean ($\bar{X}$) is 186.1 mm and the standard deviation is 12.7 mm. Use the one-sample t-test to test the hypothesis:

$$H_0 : \mu = 190 \ \text{ vs } \ H_1 : \mu < 190$$

At level of significance $\alpha = 0.05$?

Solution

- Conditions

(1) Normal Distribution.
(2) Population Standard Deviation σ is Unknown ($S = 12.7$).
(3) Sample size (n) is small ($n = 10 < 30$).

- Test Statistic Value

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{186.1 - 190}{12.7/\sqrt{10}} = -0.971$$

- Rejection Rule (One-Sided Lower Tailed Test)

Reject H_0 at significance level $\alpha = 0.05$ if $t < -t_{(n-1\,,1-\alpha)}$ otherwise Accept H_0.

- Critical Value

$$-t_{(n-1\,,1-\alpha)} = -t_{(9\,,\ 0.95)} = -1.833$$

- Decision

We get $t = -0.971 > -t_{(9\,,\ 0.95)} = -1.833$
Then Accept H_0: μ = 190 and therefore Reject H_1: μ < 190 at α= 0.05.

Question (2)

Suppose it is known that in a certain large human population cranial length is normally distributed. A random sample of size 10 is taken from this population showed that the sample mean ($\bar{X}$) is 196.1 mm and the standard deviation is 9.7 mm. Use the one-sample t-test to test the hypothesis:

$$H_0 : \mu = 190 \text{ vs } H_1 : \mu > 190$$

At level of significance $\alpha = 0.05$?

Solution

- Conditions

(1) Normal Distribution.
(2) Population Standard Deviation σ is Unknown ($S = 9.7$).
(3) Sample size (n) is small ($n = 10 < 30$).

- Test Statistic Value

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{196.1 - 190}{9.7/\sqrt{10}} = 1.989$$

- Rejection Rule (One-Sided Lower Tailed Test)

Reject H_0 at significance level $\alpha = 0.05$ if $t > t_{(n-1\,,1-\alpha)}$ otherwise Accept H_0.

- Critical Value

$$t_{(n-1\,,1-\alpha)} = t_{(9\,,\ 0.95)} = 1.833$$

- Decision

We get $t = 1.989 > t_{(9\,,\ 0.95)} = 1.833$
Then Reject H_0: μ = 190 and therefore Accept H_1: μ > 190 at α= 0.05.

Question (3)

Suppose it is known that in a certain large human population cranial length is normally distributed. A random sample of size 28 is taken from this population showed that the sample mean ($\overline{X}$) is 188.5 mm and the standard deviation is 6.7 mm. Use the one-sample t-test to test the hypothesis:

$$H_0 : \mu = 190 \ \text{ vs } \ H_1 : \mu \neq 190$$

At level of significance $\alpha = 0.05$?

Solution

- Conditions

(1) Normal Distribution.

(2) Population Standard Deviation σ is Unknown ($S = 6.7$).

(3) Sample size (n) is small ($n = 28 < 30$).

- Test Statistic Value

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{188.5 - 190}{6.7/\sqrt{28}} = -1.185$$

- Rejection Rule (One-Sided Lower Tailed Test)

Reject H_0 at significance level $\alpha = 0.05$ if $|t| > t_{(n-1\,,1-\alpha/2)}$ otherwise Accept H_0.

- Critical Value

$$t_{(n-1\,,1-\alpha/2)} = t_{(27\,,\ 0.975)} = 2.052$$

- Decision

We get $|t| = |-1.185| = 1.185 < t_{(27\,,\ 0.975)} = 2.052$

Then Accept H_0: μ = 190 and therefore Reject H_1: μ ≠ 190 at α= 0.05.

Exercises

Exercise (1)

A computer company claims that the mean time taken to learn how to use software is not more than 3 hours. A random sample of size 20 persons was selected and the data are taken shows that the sample mean is 3.23 hour and sample standard deviation is 0.51 hour. Conduct a hypothesis test using alpha = 0.05? Assume normal distribution.

Answer: H_0: $\mu = 3$ vs H_1: $\mu > 3$; Reject the null hypothesis. This means that the company's claim is true.

Exercise (2)

A random sample of 15 households from Jordan showed that they spent on average JD350 per month on food with a standard deviation of JD50. Can you conclude that the mean food expenditure is different from JD400 using alpha = 0.1? Assume normal distribution.

Answer: H_0: $\mu = 400$ vs H_1: $\mu \neq 400$; Reject the null hypothesis. This means that the mean food expenditure is different from JD400. [redacted]

Exercise (3)

Suppose we suspect that the mean height of a particular species of plant is less than the accepted mean height of 10 inches. Suppose we collect a random sample of plants with the following information:

- Sample size n = 25

- Sample mean = 9.5
- Sample standard deviation = 3.5

Test this hypothesis at significance level $\alpha = 0.05$?

Answer: We fail to reject the null hypothesis. We do not have sufficient evidence to say that the mean height for this particular plant species is less than 10 inches.
