Chapter 12

Multisample Inference

One-Way ANOVA—Fixed-Effects Model

Suppose there are k groups with n_i observations in the ith group. The jth observation in the ith group will be denoted by y_{ij}. Let us assume the following model.

$$y_{ij} = \mu + \alpha_i + e_{ij}$$

Where μ is a constant, α_i is a constant specific to the ith group, and e_{ij} is an error term, which is normally distributed with mean 0 and variance σ^2. A typical observation from the ith group is normally distributed with mean $\mu+\alpha_i$ and variance σ^2.

Table 12.1 FEF data for smoking and nonsmoking males

Group number, i	Group name	Mean FEF (L/s)	*sd* FEF (L/s)	n_i
1	NS	3.78	0.79	200
2	PS	3.30	0.77	200
3	NI	3.32	0.86	50
4	LS	3.23	0.78	200
5	MS	2.73	0.81	200
6	HS	2.59	0.82	200

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How do you compare the means of the six groups?

In a **one-way analysis of variance**, or a **one-way ANOVA** model, the means of an arbitrary number of groups, each of which follows a normal distribution with the same variance, can be compared. Whether the variability in the data comes mostly from variability within groups or can truly be attributed to variability between groups can also be determined.

Interpretation of the parameters of a one-way ANOVA fixed-effects model

1. μ represents the underlying mean of all groups taken together.
2. α_i represents the difference between the mean of the *i*th group and the overall mean.
3. e_{ij} represents random error about the mean $\mu+\alpha_i$ for an individual observation from the *i*th group.

Hypothesis Testing in One-Way ANOVA—Fixed-Effects Model

F test for overall comparison of group means

$$y_{ij} - \overline{\overline{y}} = (y_{ij} - \overline{y}_i) + (\overline{y}_i - \overline{\overline{y}})$$

$(y_{ij} - \overline{y}_i)$ represents the deviation of an individual observation from the group mean for that observation and is an indication of *within-group variability*.

$(\overline{y}_i - \overline{\overline{y}})$ represents the deviation of a group mean from the overall mean and is an indication of *between-group variability*.

Generally, if the between-group variability is large and the within-group variability is small, then H_o is rejected and the underlying group means are declared significantly different. Conversely, if the between-group variability is small and the within-group variability is large, then H_o, the hypothesis that the underlying group means are the same, is accepted.

Squared and summation of the squared deviations of the previous equation give us $$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{\bar{y}})^2=\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2+\sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_i-\bar{\bar{y}})^2$$

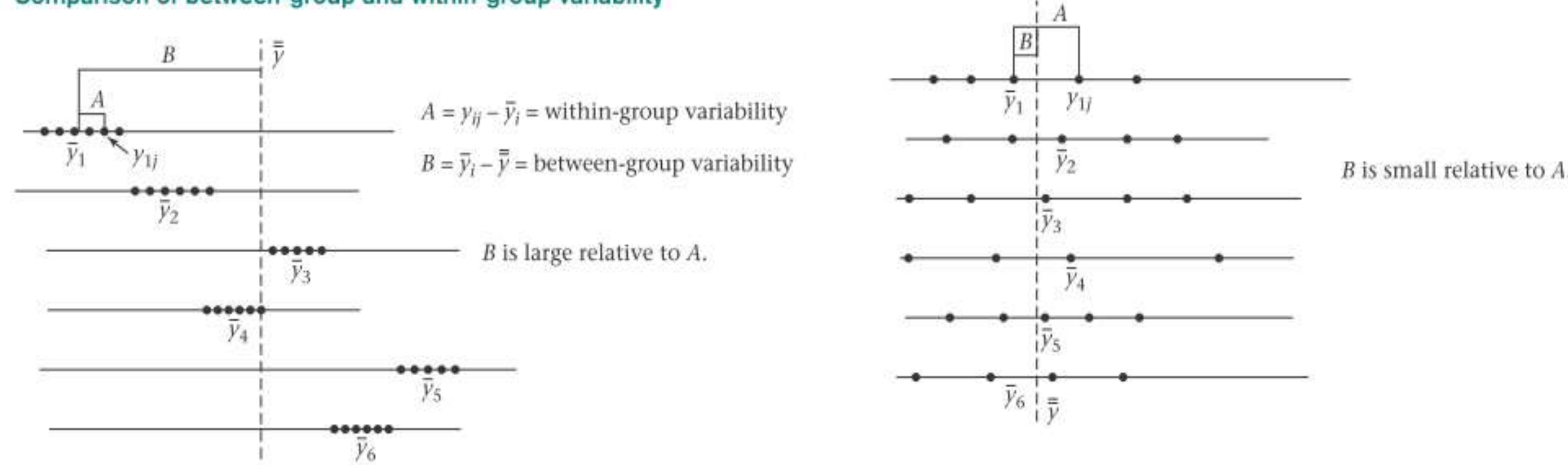
The term $\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{\bar{y}})^2$ is called the Total Sum of Squares (Total SS)

The term $\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2$ is called the Within Sum of Squares (Within SS)

The term $\sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_i-\bar{\bar{y}})^2$ is called the Between Sum of Squares (Between SS)

Thus, the relationship can be written as

Total SS = Between SS + Within SS



Comparison of between-group and within-group variability

Short computational form for the Between SS and Within SS

$$\text{Between SS} = \sum_{i=1}^{k} n_i \bar{y}_i^2 - \frac{\left(\sum_{i=1}^{k} n_i \bar{y}_i \right)^2}{n} = \sum_{i=1}^{k} n_i \bar{y}_i^2 - \frac{y_{..}^2}{n}$$

$$\text{Within SS} = \sum_{i=1}^{k} (n_i - 1) s_i^2$$

where $y_{..}$= sum of the observations across all groups, i.e., the grand total of all observations over all groups and n = total number of observations over all groups.

Between Mean Square = Between MS = Between SS/(k-1)

Within Mean Square = Within MS = Within SS/(n-k)

The significance test is based on the ratio of Between MS to Within MS. If this ratio is large, then we reject H_o; if it is small, we accept H_o. Under H_o, the ratio follows an F distribution with k - 1 and n - k degrees of freedom.

<u>Overall F test for One-way ANOVA</u>

To test the hypothesis H_o: $\alpha_i = 0$ for all i vs. H_1: at least one $\alpha_i \neq 0$, use the following procedure.

1.Compute the Between SS, Between MS, Within SS, and Within MS.

2.Compute the test statistic F = Between MS/Within MS, which follows an F distribution with k - 1 and n - k df under H_o.

If $F > F_{k-1,n-k,1-\alpha}$ then reject H_o.

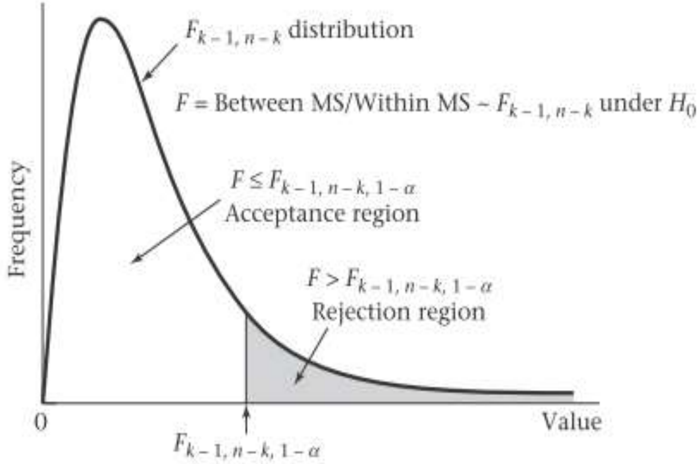
If $F \leq F_{k-1,n-k,1-\alpha}$ then accept H_o.

3.The exact p-value is given by the area to the right of F under an $F_{k-1,n-k}$ distribution = $Pr(F_{k-1,n-k} > F)$

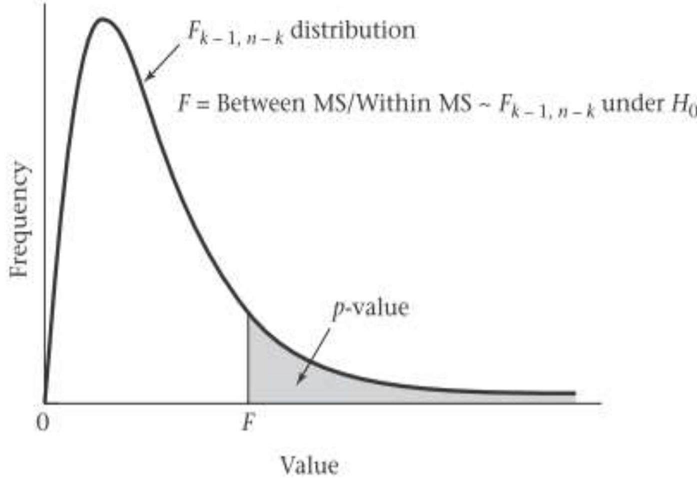
Display of one-way ANOVA results

Source of variation	SS	df	MS	F statistic	p-value
Between	$\sum_{i=1}^{k} n_i \bar{y}_i^2 - \frac{y_{..}^2}{n} = A$	$k-1$	$\frac{A}{k-1}$	$\frac{A/(k-1)}{B/(n-k)} = F$	$Pr(F_{k-1,n-k} > F)$
Within	$\sum_{i=1}^{k} (n_i - 1)s_i^2 = B$	$n-k$	$\frac{B}{n-k}$		
Total	Between SS + Within SS				

Acceptance and rejection regions for the overall F test for one-way ANOVA



Computation of the exact p-value for the overall F test for one-way ANOVA



ANOVA table for FEF data in Table 12.1

	SS	*df*	MS	*F* statistic	*p*-value
Between	184.38	5	36.875	58.0	$p < .001$
Within	663.87	1044	0.636		
Total	848.25				

To test whether the mean FEF scores differ significantly among the six groups

1. Calculate the Between MS and Within MS.
2. Determine F = Between MS/Within MS
3. Find the correlating F value from Table 9 in the Appendix
4. If $p < 0.001$, then we can reject H_0, that is, all means are equal and can conclude that at least two of the means are significantly different.

Comparisons of Specific Groups in One-Way ANOVA

H_o: all group means are equal vs. H_1: at least two group means are different. This test lets us detect when at least two groups have different underlying means, but it does not let us determine which of the groups have means that differ from each other. The usual practice is to perform the overall F test. If H_o is rejected, the specific groups are compared.

<u>*t* Test for Comparison of Pairs of Groups</u>

If we want to test whether groups 1 and 2 have means that are significantly different from each other. Under either hypothesis,

$\overline{Y}_1$ is normally distributed with mean $\mu+\alpha_1$ and variance σ^2/n_1

And $\overline{Y}_2$ is normally distributed with mean $\mu+\alpha_2$ and variance σ^2/n_2

The difference of the sample means $(\overline{y}_1 - \overline{y}_2)$ will be used as a test criterion

$$\overline{Y}_1 - \overline{Y}_2 \sim N\left[\alpha_1 - \alpha_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right]$$

This reduces to

$$\overline{Y}_1 - \overline{Y}_2 \sim N\left[0, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right]$$

If σ^2 were known, dividing by the standard error we get

$$Z = \frac{\overline{Y}_1 - \overline{Y}_2}{\sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

The test statistic Z would follow an $N(0,1)$ distribution under H_0. Because σ^2 is generally unknown, the best estimate of it, denoted by s^2, is substituted and the statistic revised accordingly.

$$S^2 = [(n_1-1)s_1^2 + (n_2-1)s_2^2]/(n_1+n_2-2)$$

For the one-way ANOVA, there are k sample variances and a similar approach is used to estimate σ^2 by computing a weighted average of k individual sample variances, where the weights are the number of degrees of freedom in each of the k samples.

Pooled estimate of the variance for one-way ANOVA

$$s^2 = \sum_{i=1}^{k}(n_i-1)s_i^2 \Big/ \sum_{i=1}^{k}(n_i-1) = \left[\sum_{i=1}^{k}(n_i-1)s_i^2\right] \Big/ (n-k) = \text{Within MS}$$

t Test for the Comparison of Pairs of Groups in One-Way ANOVA (LSD Procedure)

Suppose we wish to compare two specific groups, arbitrarily labeled as group 1 and group 2, among k groups. To test the hypothesis $H_0: \alpha_1 = \alpha_2$ vs. $H_1: \alpha_1 \neq \alpha_2$, use the following procedure

1.Compute the pooled estimate of the variance s^2 = within MS from the one way ANOVA

2.Compute the test statistic $t = \dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{s^2\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$ which follows a t_{n-k} distribution under H_0.

3.For a two-sided level α test, if $t > t_{n-k,1-\alpha/2}$ or $t < t_{n-k,\alpha/2}$ then reject H_0.

$t_{n-k,\alpha/2} \leq t \leq t_{n-k,1-\alpha/2}$ then accept H_0.

1.The exact p-value is given by

$p = 2 \times$ the area to the left of t under a t_{n-k} distribution if $t < 0$

$= 2 \times Pr(t_{n-k} < t)$

$p = 2 \times$ the areas to the right of t under a t_{n-k} distribution if $t \geq 0$

$= 2 \times Pr(t_{n-k} > t)$

This test is often referred to as the *least significant difference (LSD) method*.

t Test for the Comparison of Pairs of Groups in One-Way ANOVA (LSD Procedure)

Suppose we wish to compare two specific groups, arbitrarily labeled as group 1 and group 2, among k groups. To test the hypothesis $H_0: \alpha_1 = \alpha_2$ vs. $H_1: \alpha_1 \neq \alpha_2$, use the following procedure:

(1) Compute the pooled estimate of the variance s^2 = Within MS from the one way ANOVA.

(2) Compute the test statistic

$$t = \frac{\overline{y}_1 - \overline{y}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

which follows a t_{n-k} distribution under H_0.

(3) For a two-sided level α test,

if $t > t_{n-k,1-\alpha/2}$ or $t < t_{n-k,\alpha/2}$

then reject H_0

if $t_{n-k,\alpha/2} \leq t \leq t_{n-k,1-\alpha/2}$

then accept H_0

(4) The exact p-value is given by

$p = 2 \times$ the area to the left of t under a t_{n-k} distribution if $t < 0$
$= 2 \times Pr(t_{n-k} < t)$

$p = 2 \times$ the area to the right of t under a t_{n-k} distribution if $t \geq 0$
$= 2 \times Pr(t_{n-k} > t)$

(5) A $100\% \times (1 - \alpha)$ CI for $\mu_1 - \mu_2$ is given by

$$\overline{y}_1 - \overline{y}_2 \pm t_{n-k,1-\alpha/2}\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

FEF data for smoking and nonsmoking males

Group number, i	Group name	Mean FEF (L/s)	*sd* FEF (L/s)	n_i
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3	NI	3.32	0.86	50
4	LS	3.23	0.78	200
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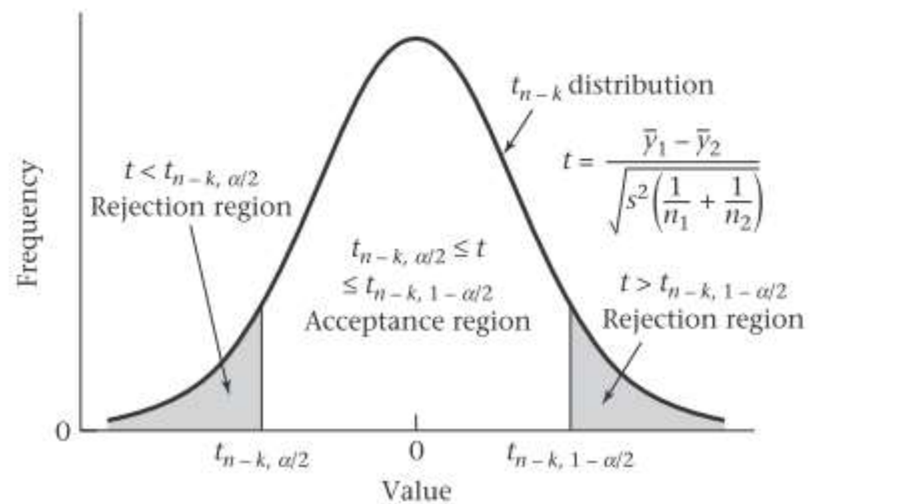
6	HS	2.59	0.82	200

Acceptance and rejection regions for the t test for the comparison of pairs of groups in one-way ANOVA (LSD approach)



Computation of the exact p-value for the t test for the comparison of pairs of groups in one-way ANOVA (LSD approach)

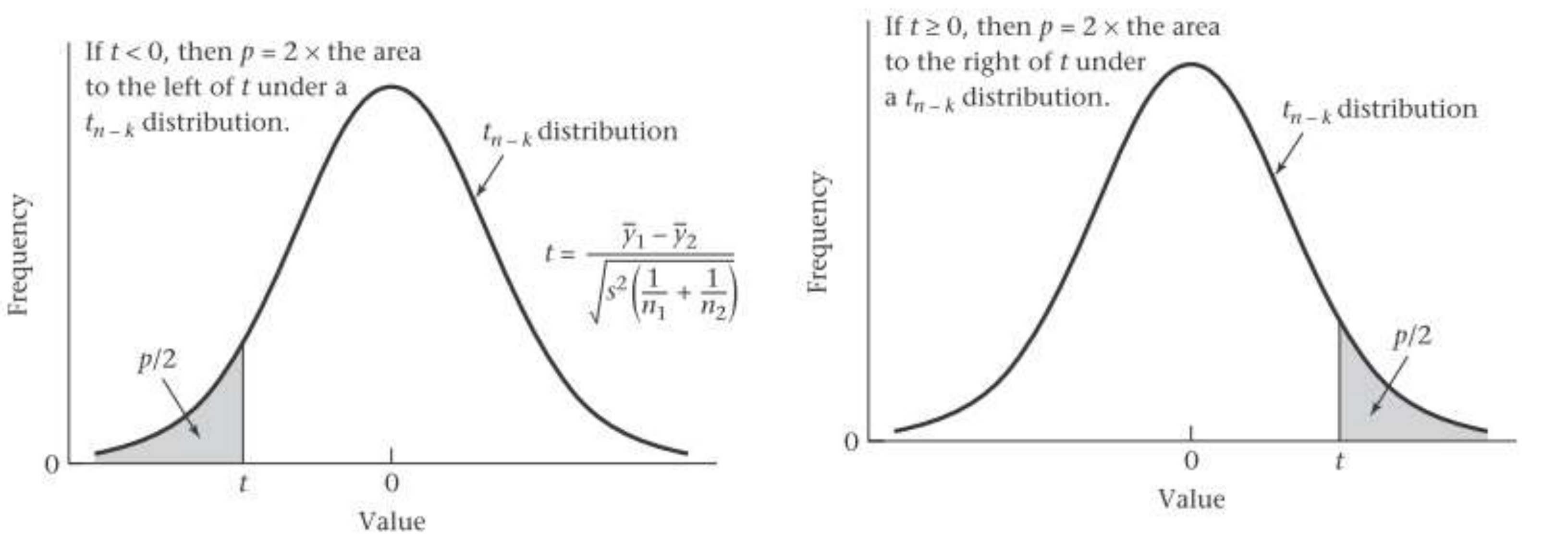
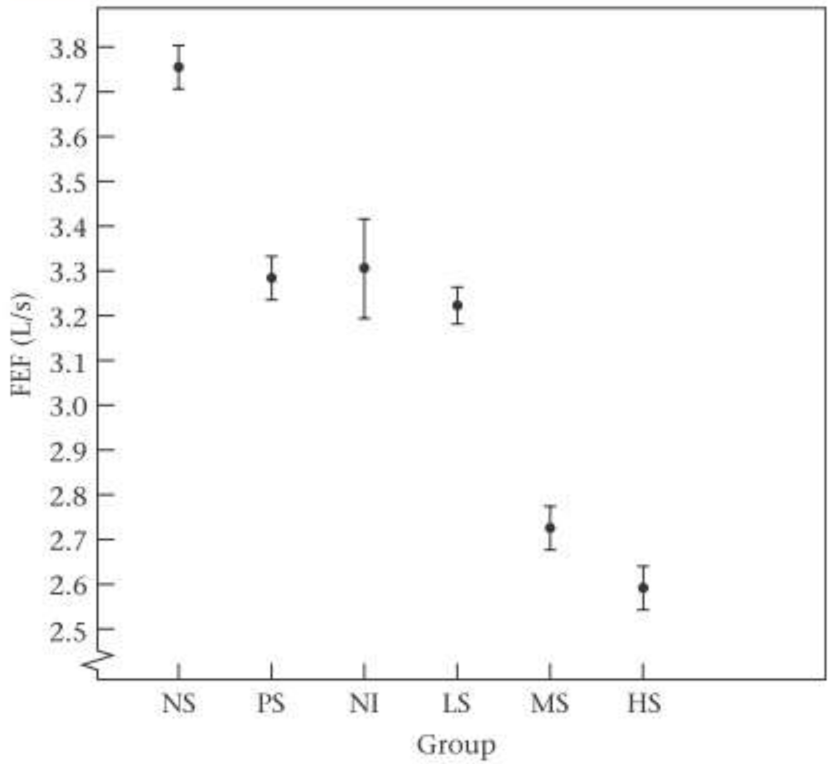


Figure 12.6 Mean ± *se* for FEF for each of six smoking groups



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A frequent error in performing the *t* test when comparing groups 1 and 2 is to use the sample variances from *these two groups* rather than from *all k groups* to estimate σ^2. If the sample variances from only two groups are used, then different estimates of σ^2 are obtained for each pair of groups considered, which is not reasonable because all the groups are assumed to have the same underlying variance σ^2.

Table 12.4 Comparisons of specific pairs of groups for the FEF data in Table 12.1 using the LSD t test approach

Groups compared	Test statistic	p-value
NS, PS	$t=\frac{3.78-3.30}{\sqrt{0.636\left(\frac{1}{200}+\frac{1}{200}\right)}}=\frac{0.48}{0.08}=6.02^{a}$	< .001
NS, NI	$t=\frac{3.78-3.32}{\sqrt{0.636\left(\frac{1}{200}+\frac{1}{50}\right)}}=\frac{0.46}{0.126}=3.65$	< .001
NS, LS	$t=\frac{3.78-3.23}{\sqrt{0.636\left(\frac{1}{200}+\frac{1}{200}\right)}}=\frac{0.55}{0.08}=6.90$	< .001
NS, MS	$t=\frac{3.78-2.73}{0.080}=\frac{1.05}{0.08}=13.17$	< .001
NS, HS	$t=\frac{3.78-2.59}{0.080}=\frac{1.19}{0.08}=14.92$	< .001
PS, NI	$t=\frac{3.30-3.32}{0.126}=\frac{-0.02}{0.126}=-0.16$	NS
PS, LS	$t=\frac{3.30-3.23}{0.080}=\frac{0.07}{0.08}=0.88$	NS
PS, MS	$t=\frac{3.30-2.73}{0.080}=\frac{0.57}{0.08}=7.15$	< .001
PS, HS	$t=\frac{3.30-2.59}{0.080}=\frac{0.71}{0.08}=8.90$	< .001
NI, LS	$t=\frac{3.32-3.23}{0.126}=\frac{0.09}{0.126}=0.71$	NS
NI, MS	$t=\frac{3.32-2.73}{0.126}=\frac{0.59}{0.126}=4.68$	< .001
NI, HS	$t=\frac{3.32-2.59}{0.126}=\frac{0.73}{0.126}=5.79$	< .001
LS, MS	$t=\frac{3.23-2.73}{0.08}=\frac{0.50}{0.08}=6.27$	< .001
LS, HS	$t=\frac{3.23-2.59}{0.08}=\frac{0.64}{0.08}=8.03$	< .001
MS, HS	$t=\frac{2.73-2.59}{0.08}=\frac{0.14}{0.08}=1.76$	NS

[a] All test statistics follow a t_{1044} distribution under H_0.

Figure 12.11 Box plots of MAXFWT by group

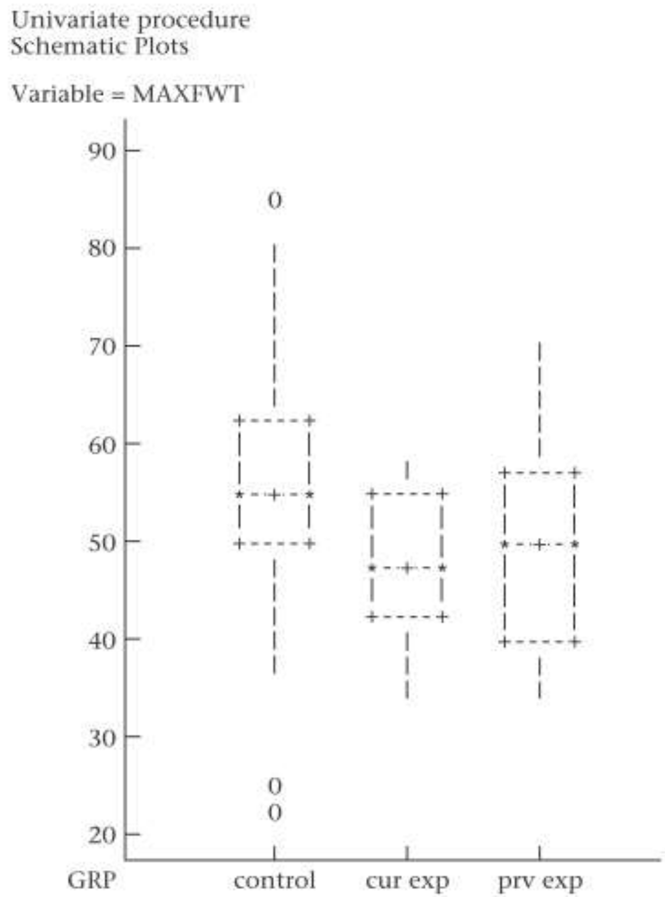

Table 12.8 Overall *F* test for one-way ANOVA for MAXFWT

GLM Procedure

Dependent Variable: MAXFWT

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	966.79062	483.39531	4.60	0.0125
Error	92	9671.14622	105.12115		
Corrected Total	94	10637.93684			

TABLE 12.9 Comparison of group means for MAXFWT for pairs of specific groups (LSD procedure)

The GLM Procedure
Least Squares Means

lead_ group	maxfwt LSMEAN	Standard Error	Pr > |t|	LSMEAN Number
Control	55.0952381	1.2917390	<.0001	1
Cur exp	47.5882353	2.4866840	<.0001	2
Prv exp	49.4000000	2.6472773	<.0001	3

Least Squares Means for lead_group
t for H0: LSMean(i)= LSMean(j) / Pr > |t|

Dependent Variable: maxfwt

i/j	1	2	3
1		2.678991	-1.933461
		0.0087	0.0563
2	-2.67899		-0.49883
	0.0087		0.6191
3	-1.93346	0.498829	
	0.0563	0.6191	