# Chapter 12

# Multisample Inference

## Fundamentals of Biostatistics

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## 12.1 Introduction to the One-Way Analysis of Variance

In Chapter 8 we were concerned with comparing the means of two normal distributions using the two-sample t test for independent samples. Frequently, the means of more than two distributions need to be compared. For example, in Pulmonary Disease, the researchers may be interested in comparing the mean of forced mid-expiratory flow (FEF) among six groups of people (*Nonsmokers*, *Passive smokers*, *Noninhaling smokers*, *Light smokers*, *Moderate smokers* and *Heavy smokers*). Therefore, the t test methodology generalizes nicely in this case to a procedure called the *one-way analysis of variance* (*ANOVA*).

Question: How can the means of these 6 groups can be compared?

## 12.2 One-Way ANOVA-Fixed-Effects Model

Suppose there are $k$ groups with $n_i$ observations in the $i^{th}$ group. The $j^{th}$ observation in the $i^{th}$ group will be denoted by $y_{ij}$. Let's assume the following model (*Fixed-Effects Model*):

**EQUATION 12.1**

$$y_{ij} = \mu + \alpha_i + e_{ij}$$

where μ is a constant, $\alpha_i$ is a constant specific to the $i^{th}$ group, and $e_{ij}$ is an error term, which is normally distributed with mean 0 and variance $\sigma^2$. Thus, a typical observation from the $i^{th}$ group is normally distributed with mean $\mu + \alpha_i$ and variance $\sigma^2$. The parameters in Equation 12.1 can be interpreted as follows:

**EQUATION 12.2**

**Interpretation of the Parameters of a One-Way ANOVA Fixed-Effects Model**

**(1)** μ represents the underlying mean of all groups taken together.

**(2)** $\alpha_i$ represents the difference between the mean of the *i*th group and the overall mean.

**(3)** $e_{ij}$ represents random error about the mean $\mu + \alpha_i$ for an individual observation from the *i*th group.

### Some Notations

- It is not possible to estimate both the overall constant μ as well as the $k$ constants $\alpha_i$, which are specific to each group. The reason is that we only have $k$ observed mean values for the $k$ groups, which are used to estimate $k+1$ parameters.
- We need to constrain the parameters so that only $k$ parameters will be estimated.
- Some typical constraints are:
  (1) the sum of the $\alpha_i$'s is set to 0, or
  (2) the $\alpha_i$ for the last group ($\alpha k$ ) is set to 0.

In this text, we will use the former approach in our Fixed-Effects Model and the Group Means are compared within the context of this model.

> **DEFINITION 12.1**
>
> The model in Equation 12.1 is a **one-way analysis of variance**, or a **one-way ANOVA model**. With this model, the means of an arbitrary number of groups, each of which follows a normal distribution with the same variance, can be compared. Whether the variability in the data comes mostly from variability within groups or can truly be attributed to variability between groups can also be determined.

### 12.3 Hypothesis Testing in One-Way ANOVA-Fixed-Effects Model

The null hypothesis ($H_0$) and the alternative hypothesis ($H_1$) can be stated as follows:

- The null hypothesis ($H_0$) in this case is that *the underlying mean of each of the $k$ groups is the same*. This hypothesis is equivalent to stating that each $\alpha_i = 0$ because the $\alpha_i$ sum up to 0 (*that is*: $H_0$: all $\alpha_i = 0$).

- The alternative hypothesis ($H_1$) is that *at least two of the group means are not the same*. This hypothesis is equivalent to stating that at least one $\alpha_i \neq 0$ (*that is*: $H_1$: at least one $\alpha_i \neq 0$).

Thus, we wish to test the hypothesis:

$$\boxed{H_0\text{: all } \alpha_i = 0 \quad \textit{versus} \quad H_1\text{: at least one } \alpha_i \neq 0}$$

### 12.3.1 F Test for Overall Comparison of Group Means

The mean for the $i^{th}$ group will be denoted by $\bar{y}_i$, and the mean over all groups by $\bar{\bar{y}}$. The deviation of an individual observation ($y_{ij}$) from the overall mean ($\bar{\bar{y}}$), that is, $(y_{ij} - \bar{\bar{y}})$, can be represented as follows:

**EQUATION 12.3**

$$y_{ij} - \bar{\bar{y}} = (y_{ij} - \bar{y}_i) + (\bar{y}_i - \bar{\bar{y}})$$

where:

(i) $(y_{ij} - \bar{y}_i)$: represents the deviation of an individual observation ($y_{ij}$) from the group mean ($\bar{y}_i$) for that observation and is an indication of *within-group variability*.

(ii) $(\bar{y}_i - \bar{\bar{y}})$: represents the deviation of a group mean ($\bar{y}_i$) from the overall mean ($\bar{\bar{y}}$) and is an indication of *between-group variability*.

**Important Notations**

- If the *between-group variability* is large and the *within-group variability* is small, then $H_0$ is rejected and the underlying group means are declared significantly different.
- If the *between-group variability* is small and the *within-group variability* is large, then $H_0$, *the hypothesis that the underlying group means are the same*, is accepted.

Now, if both sides of Equation 12.3 are squared and the squared deviations are summed over all observations over all groups, then the following relationship is obtained:

**EQUATION 12.4**

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \bar{\bar{y}})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_i - \bar{\bar{y}})^2$$

where the *cross-product term* can be shown to be zero. Thus, this will leads to three important definitions given as follows:

## Important Definitions

**DEFINITION 12.2** The term

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij}-\bar{\bar{y}}\right)^2$$

is called the **Total Sum of Squares (Total SS).**

**DEFINITION 12.3** The term

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij}-\bar{y}_i\right)^2$$

is called the **Within Sum of Squares (Within SS).**

**DEFINITION 12.4** The term

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\bar{y}_i-\bar{\bar{y}}\right)^2$$

is called the **Between Sum of Squares (Between SS).**

Thus, the relationship in Equation 12.4 can be written as follows:

$$\boxed{\begin{gathered}Total\ SS \ = \ Within\ SS \ + \ Between\ SS\\ SST \ = \ SSW \ + \ SSB\end{gathered}}$$

To perform the hypothesis test, it is easier to use the short computational form for the Within SS and Between SS in Equation 12.5 as follows:

**EQUATION 12.5**

**Short Computational Form for the Between SS and Within SS**

$$\text{Between SS} = \sum_{i=1}^{k} n_i\bar{y}_i^2 - \frac{\left(\sum_{i=1}^{k} n_i\bar{y}_i\right)^2}{n} = \sum_{i=1}^{k} n_i\bar{y}_i^2 - \frac{y_{..}^2}{n}$$

$$\text{Within SS} = \sum_{i=1}^{k}(n_i-1)s_i^2$$

where $y_{..}$ = sum of the observations across all groups, $n$ = total number of observations over all groups, and $s_i^2$ = sample variance for the $i$th group.

Finally, the following two definitions are also important:

**DEFINITION 12.5** **Between Mean Square = Between MS** = Between SS/$(k - 1)$

**DEFINITION 12.6** **Within Mean Square = Within MS** = Within SS/$(n - k)$

The significance test will be based on the ratio of the **Between MS** to the **Within MS**. Then:

- If the ratio is large, then we reject $H_0$.
- If the ratio is small, we accept (or *fail to reject*) $H_0$.

**Notation:** Under $H_0$, the ratio of **Between MS** to **Within MS** follows an **F-distribution** with degrees of freedom $(k-1)$ and $(n-k)$. Thus, the following **test procedure** for a significance level α test is used.

**EQUATION 12.6**

**Overall *F* Test for One-Way ANOVA**

To test the hypothesis $H_0$: $\alpha_i = 0$ for all $i$ vs. $H_1$: at least one $\alpha_i \neq 0$, use the following procedure:

(1) Compute the Between SS, Between MS, Within SS, and Within MS using Equation 12.5 and Definitions 12.5 and 12.6.

(2) Compute the test statistic $F$ = Between MS/Within MS, which follows an $F$ distribution with $k - 1$ and $n - k$ *df* under $H_0$.

(3) If $F > F_{k-1,n-k,1-\alpha}$ then reject $H_0$
If $F \le F_{k-1,n-k,1-\alpha}$ then accept $H_0$

(4) The exact $p$-value is given by the area to the right of $F$ under an $F_{k-1,n-k}$ distribution $= Pr(F_{k-1,n-k} > F)$.

The acceptance and rejection regions for this test are shown in Figure 12.2. Computation of the exact $p$-value is illustrated in Figure 12.3. The results from the ANOVA are typically displayed in an ANOVA table, as in Table 12.2.

**TABLE 12.2** **Display of one-way ANOVA results**

| Source of variation | SS | *df* | MS | *F* statistic | *p*-value |
|---|---|---|---|---|---|
| Between | $\sum_{i=1}^{k} n_i \bar{y}_i^2 - \frac{y_{..}^2}{n} = B$ | $k-1$ | $\frac{B}{k-1}$ | $\frac{B/(k-1)}{A/(n-k)} = F$ | $Pr(F_{k-1,n-k} > F)$ |
| Within | $\sum_{i=1}^{k} (n_i - 1)s_i^2 = A$ | $n-k$ | $\frac{A}{n-k}$ | | |
| Total | Between SS + Within SS | | | | |

**FIGURE 12.2** Acceptance and rejection regions for the overall $F$ test for one-way ANOVA

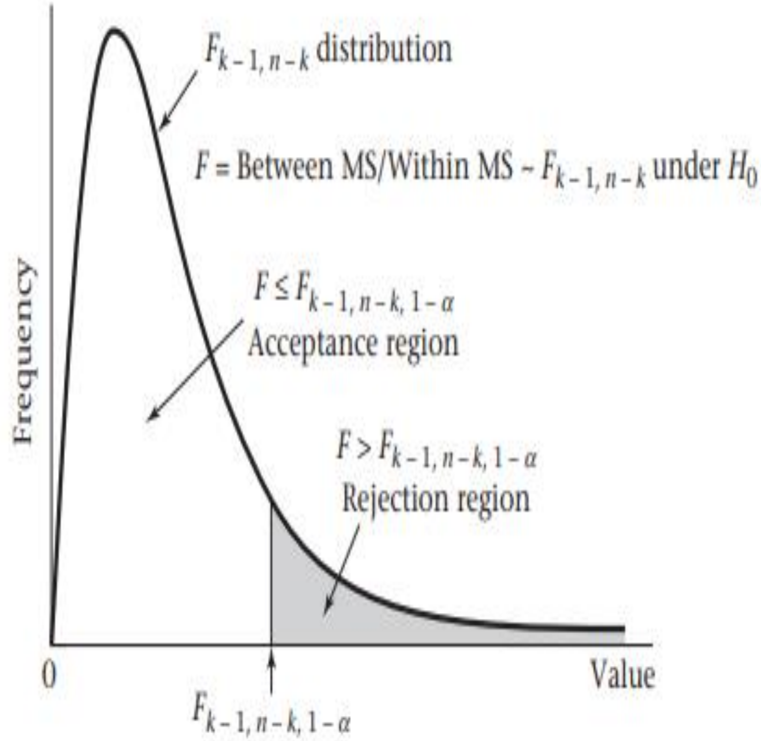
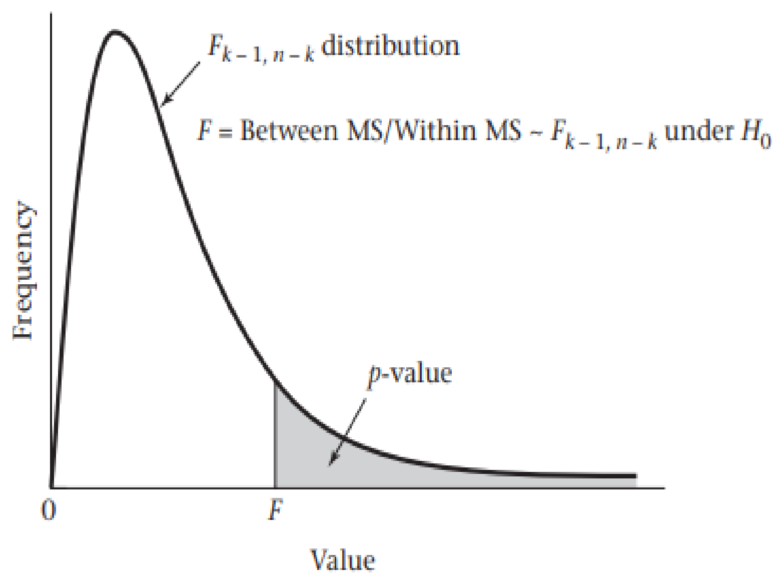



**FIGURE 12.3** Computation of the exact $p$-value for the overall $F$ test for one-way ANOVA



**EXAMPLE 12.1**

Pulmonary Disease A topic of public-health interest is whether passive smoking (*exposure among nonsmokers to cigarette smoke in the atmosphere*) has a measurable effect on pulmonary health. Researchers studied this question by measuring pulmonary function in several ways in the following $(k = 6)$ groups:

(1) Nonsmokers (NS): People who themselves did not smoke and were not exposed to cigarette smoke either at home or on the job.

(2) Passive smokers (PS): People who themselves did not smoke and were not exposed to cigarette smoke in the home but were employed for 20 or more years in an enclosed working area that routinely contained tobacco smoke.

(3) Noninhaling smokers (NI): People who smoked pipes, cigars, or cigarettes but who did not inhale.

(4) Light smokers (LS): People who smoked and inhaled 1–10 cigarettes per day for 20 or more years. (*Note: There are 20 cigarettes in a pack.*)

(5) Moderate smokers (MS): People who smoked and inhaled 11–39 cigarettes per day for 20 or more years.

(6) Heavy smokers (HS): People who smoked and inhaled 40 or more cigarettes per day for 20 or more years.

A principal measure used by researchers to assess pulmonary function was forced mid-expiratory flow (FEF). They were interested in comparing mean FEF among the six groups.

**Notation**

- **Forced Expiration (FE)** is a simple but extremely useful pulmonary function test. A spirometry tracing is obtained by having a person inhale to total lung capacity and then exhaling as hard and as completely as possible.

- **Forced Expiratory Flow (FEF)** is the flow (*or speed*) of air coming out of the lung during the middle portion of a forced expiration.

The researchers identified 200 males and 200 females in each of the 6 groups except for the NI group, which was limited to 50 males and 50 females because of the small number of such people available. The mean and standard deviation of FEF for each of the 6 groups for males are presented in Table 12.1:

**TABLE 12.1** FEF data for smoking and nonsmoking males

| Group number, $i$ | Group name | Mean FEF (L/s) | *sd* FEF (L/s) | $n_i$ |
|---|---|---|---|---|
| 1 | NS | 3.78 | 0.79 | 200 |
| 2 | PS | 3.30 | 0.77 | 200 |
| 3 | NI | 3.32 | 0.86 | 50 |
| 4 | LS | 3.23 | 0.78 | 200 |
| 5 | MS | 2.73 | 0.81 | 200 |
| 6 | HS | 2.59 | 0.82 | 200 |

## Notation

Intuitively, in Table 12.1 an observed FEF is represented as a sum of an overall mean FEF plus an effect of each smoking group plus random variability within each smoking group.

Answer the following:

(a) Compute the Within SS and Between SS for the FEF data in Table 12.1?

**Solution:** We use Equation 12.5 as follows:

Step (1): The total number of observations over all groups ($n$) can be calculated as follows:

$$n = \sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k$$

$$\begin{aligned} n &= \sum_{i=1}^{6} n_i = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \\ &= 200 + 200 + 50 + 200 + 200 + 200 \\ &= 1050 \end{aligned}$$

Step (2): The mean for the $i^{th}$ group ($\bar{y}_i$) and the sum of the observations across all groups ($y_{..}$) can be calculated as follows:

$$\bar{y}_i = \frac{\sum_{j=1}^{n_i} y_{ij}}{n_i} \longrightarrow \sum_{j=1}^{n_i} y_{ij} = n_i * \bar{y}_i$$

Thus ($y_{..}$) can be calculated as follows:

$$y_{..} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^{k} n_i * \bar{y}_i$$

Then

$$\begin{aligned} y_{..} &= \sum_{i=1}^{6} \sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^{6} n_i * \bar{y}_i \\ &= n_1 * \bar{y}_1 + n_2 * \bar{y}_2 + n_3 * \bar{y}_3 + n_4 * \bar{y}_4 + n_5 * \bar{y}_5 + n_6 * \bar{y}_6 \\ &= (200)(3.78) + (200)(3.30) + (50)(3.32) + (200)(3.23) \\ &\qquad + (200)(2.73) + (200)(2.59) \\ &= 3292 \end{aligned}$$

Step (3): The Between Sum of Squares (Between SS) can be calculated as follows:

$$\text{Between SS} = \sum_{i=1}^{k=6} n_i \bar{y}_i^2 - \frac{y_{..}^2}{n}$$

$$\text{Between SS} = \left[200(3.78)^2 + 200(3.30)^2 + \cdots + 200(2.59)^2\right]$$
$$- \frac{\left[200(3.78) + 200(3.30) + \cdots + 200(2.59)\right]^2}{1050}$$
$$= 10,505.58 - 3292^2/1050 = 10,505.58 - 10,321.20 = 184.38$$

Step (4): The Within Sum of Squares (Within SS) can be calculated as follows:

$$\text{Within SS} = \sum_{i=1}^{k=6} (n_i - 1)\, s_i^2$$
$$= (n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2 + \cdots + (n_6 - 1)\, s_6^2$$
$$\text{Within SS} = 199(0.79)^2 + 199(0.77)^2 + 49(0.86)^2 + 199(0.78)^2$$
$$+ 199(0.81)^2 + 199(0.82)^2$$
$$= 124.20 + 117.99 + 36.24 + 121.07 + 130.56 + 133.81 = 663.87$$

(b) Test whether the mean FEF scores differ significantly among the six groups $(k = 6)$ in Table 12.1? Use $\alpha = 0.05$?

**Solution**

Step (1): First, we write (*state*) the hypotheses $H_0$ and $H_1$ as follows:

$H_0$: The underlying mean FEF of each of the 6 groups is the same.

*versus* $H_1$: At least two of the group means are not the same.

$\boxed{H_0: \text{all } \alpha_i = 0 \quad \textit{versus} \quad H_1: \text{at least one } \alpha_i \neq 0 \quad ; \quad i = 1, 2, 3, 4, 5, 6}$

Step (2): The test will be based on the ratio (F-value) of the Between MS to the Within MS which can be calculated as follows:

From part (a) of this example, and because there are $n = 1050$ observations combined over all 6 groups, we have:

(1) Between SS = 184.38, then:

Between MS = Between SS $/(k-1)$

= 184.38 / 5 = 36.875

(2) Within SS = 663.87, then:

Within MS = Within SS $/(n-k)$

= 663.87 / (1050 - 6)

= 663.87 / 1044 = 0.636

(3) Test Statistic value (calculated $F - value$) is given as follows:

F = Between MS / Within MS

= 36.875 / 0.636

= 58.00 $\sim F_{(k-1,\ n-k,\ 1-\alpha)}$ under $H_0$.

(4) Critical value

To find the critical value, we use the F-table (*Table 8-Percentage points of the F distribution* ($F_{d1,\, d2,\, p}$)) in the Appendix page 882-883. A part of this table is shown below:

**TABLE 8** Percentage points of the *F* distribution ($F_{d_1,d_2,p}$)

| *df* for denominator, $d_2$ | $p$ | *df* for numerator, $d_1$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .90 | 39.86 | 49.50 | 53.59 | 55.83 | 57.24 | 58.20 | 58.91 | 59.44 | 60.71 | 62.00 | 63.33 |
| | .95 | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 236.8 | 238.9 | 243.9 | 249.1 | 254.3 |
| | .975 | 647.8 | 799.5 | 864.2 | 899.6 | 921.8 | 937.1 | 948.2 | 956.7 | 976.7 | 997.2 | 1018. |
| | .99 | 4052. | 5000. | 5403. | 5625. | 5764. | 5859. | 5928. | 5981. | 6106. | 6235. | 6366. |
| | .995 | 16211. | 20000. | 21615. | 22500. | 23056. | 23437. | 23715. | 23925. | 24426. | 24940. | 25464. |
| | .999 | 405280. | 500000. | 540380. | 562500. | 576400. | 585940. | 592870. | 598140. | 610670. | 623500. | 636620. |
| 2 | .90 | 8.53 | 9.00 | 9.16 | 9.24 | 9.29 | 9.33 | 9.35 | 9.37 | 9.41 | 9.45 | 9.49 |
| | .95 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.41 | 19.45 | 19.50 |
| | .975 | 38.51 | 39.00 | 39.17 | 39.25 | 39.30 | 39.33 | 39.36 | 39.37 | 39.42 | 39.46 | 39.50 |
| | .99 | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.37 | 99.42 | 99.46 | 99.50 |
| | .995 | 198.5 | 199.0 | 199.2 | 199.2 | 199.3 | 199.3 | 199.4 | 199.4 | 199.4 | 199.5 | 199.5 |
| | .999 | 998.5 | 999.0 | 999.2 | 999.2 | 999.3 | 999.3 | 999.4 | 999.4 | 999.4 | 999.5 | 999.5 |

Thus

$$\begin{aligned} F_{(k-1,\ n-k,\ 1-\alpha)} &= F_{(6-1,\ 1050-6,\ 1-0.05)} \\ &= F_{(5,\ 1044,\ 0.95)} \\ &= 2.22 \end{aligned}$$

Note that: The critical value is obtained from the F-table using MINITAB.

(5) One-Way ANOVA Table

The results obtained in (1) – (3) are displayed in an ANOVA table (*One-Way ANOVA Table*) (Table 12.3) which is shown below:

**TABLE 12.3** ANOVA table for FEF data in Table 12.1

| | SS | *df* | MS | *F* statistic | *p*-value |
|---|---|---|---|---|---|
| Between | 184.38 | 5 | 36.875 | 58.0 | $p < .001$ |
| Within | 663.87 | 1044 | 0.636 | | |
| Total | 848.25 | | | | |

(6) Rule, Decision and Conclusion

Because:

$$F = 58.00 > F_{(5,\ 1044,\ 0.95)} = 2.22$$

Then, we can reject $H_0$ at $\alpha = 0.05$, *that all the means are equal*, and accept $H_1$, and conclude that *at least two of the means are significantly different*.

Note that

The exact $p - value$ is given by the area to the right of F under an $F_{(k-1,\ n-k,\ 1-\alpha)}$ distribution, that is:

$$\begin{aligned} p - value &= P\left(F_{(k-1,\ n-k,\ 1-\alpha)} > F\right) \\ &= P\left(F_{(5,\ 1044,\ 0.95)} > 58.00\right) \\ &\approx 2.5 \times 10^{-53} < \alpha = 0.05 \end{aligned}$$

Then, at $\alpha = 0.05$ we can reject $H_0$ and accept $H_1$, and conclude that *at least two of the means are significantly different*.

### 12.4 Comparisons of Specific Groups in One-Way ANOVA

In the previous section (Section 12.3) a test of the hypothesis $H_0$: all group means are equal, versus $H_1$: at least two group means are different, was presented. **This test lets us detect when at least two groups have different underlying means, but it does not let us determine which of the groups have means that differ from each other**. The usual practice is to perform the overall F test just discussed. If $H_0$ is rejected, then specific groups are compared, as discussed in this section.

#### 12.4.1 The t-Test for Comparison of Pairs of Groups

Suppose at this point we want to test whether groups 1 and 2 have means that are significantly different from each other. From the underlying model in Equation 12.1, under either hypothesis, we have:

**Equation 12.7**

$\bar{Y}_1$ is normally distributed with mean $\mu + \alpha_1$ and variance $\sigma^2/n_1$

and $\bar{Y}_2$ is normally distributed with mean $\mu + \alpha_2$ and variance $\sigma^2/n_2$

The difference of the sample means $(\bar{y}_1 - \bar{y}_2)$ will be used as a test criterion. Thus, from Equation 12.7, because the samples are independent it follows that:

**Equation 12.8**

$$\bar{Y}_1 - \bar{Y}_2 \sim N\left[\alpha_1 - \alpha_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right]$$

However, under $H_0$: $\alpha_1 = \alpha_2$ and Equation 12.8 reduces to the following:

**Equation 12.9**

$$\bar{Y}_1 - \bar{Y}_2 \sim N\left[0, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right]$$

If $\sigma^2$ were known, then we could divide by the standard error $\left(\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$ to obtain the following test statistic:

**Equation 12.10**

$$Z = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

**Notation**

The test statistic Z would follow an N(0, 1) distribution under $H_0$. Because $\sigma^2$ is generally unknown, the best estimate of it, denoted by $s^2$, is substituted, and the test statistic is revised accordingly.

Question: How should $\sigma^2$ be estimated?

Answer: The pooled estimate of group-specific variances is reasonable to estimate $\sigma^2$. In particular, from Equation 8.10, we have this estimator is given as follows:

**Equation 8.10**

$$s^2 = \left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right] / (n_1 + n_2 - 2)$$

But, the one-way ANOVA, there are $k$ sample variances and a similar approach is used to estimate $\sigma^2$ by computing a weighted average of $k$ individual sample variances, where the weights are the number of degrees of freedom in each of the $k$ samples. This formula is given as follows:

**Equation 12.11**

**Pooled Estimate of the Variance for One-Way ANOVA**

$$s^2 = \sum_{i=1}^{k}(n_i - 1)s_i^2 \Big/ \sum_{i=1}^{k}(n_i - 1) = \left[\sum_{i=1}^{k}(n_i - 1)s_i^2\right] \Big/ (n-k) = \text{Within MS}$$

However, note from Equations 12.5, 12.11, and Definition 12.6 (*on page 555*) that this weighted average is the same as the Within MS. Thus, the Within MS is used to estimate $\sigma^2$. The pooled estimate of the variance, that is, $s^2$, for the one-way ANOVA, has the following number of degrees of freedom ($df$):

$$(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) df = (n_1 + n_2 + \cdots + n_k) - k = n - k\, df$$

**EXAMPLE 12.6**

Pulmonary Disease For the FEF data in Table 12.1, answer the following:

(a) What is the best estimate of $\sigma^2$?

(b) How many degrees of freedom ($df$) does it have?

Solution

From Table 12.3, we have:

(a) The best estimate of the variance ($\sigma^2$) is the:

$\text{Within MS} = 0.636.$

(b) The number of degrees of freedom ($df$) is:

$$\begin{aligned} df &= n - k \\ &= 1050 - 6 \\ &= 1044. \end{aligned}$$

Hence, the test statistic Z in Equation 12.10 will be revised, substituting $s^2$ for $\sigma^2$, with the new test statistic t distributed as $t_{(n-k)}$ rather than $N(0, 1)$.

This test is often referred to as the least significant difference (LSD) method. The test procedure (*Least Significant Difference (LSD)*) is given as follows:

## Equation 12.12

**$t$ Test for the Comparison of Pairs of Groups in One-Way ANOVA (LSD Procedure)**

Suppose we wish to compare two specific groups, arbitrarily labeled as group 1 and group 2, among $k$ groups. To test the hypothesis $H_0: \alpha_1 = \alpha_2$ vs. $H_1: \alpha_1 \neq \alpha_2$, use the following procedure:

(1) Compute the pooled estimate of the variance $s^2$ = Within MS from the one way ANOVA.

(2) Compute the test statistic

$$t = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

which follows a $t_{n-k}$ distribution under $H_0$.

(3) For a two-sided level $\alpha$ test,

if $t > t_{n-k,1-\alpha/2}$ or $t < t_{n-k,\alpha/2}$

then reject $H_0$

if $t_{n-k,\alpha/2} \le t \le t_{n-k,1-\alpha/2}$

then accept $H_0$

(4) The exact $p$-value is given by

$p = 2 \times$ the area to the left of $t$ under a $t_{n-k}$ distribution if $t < 0$
$= 2 \times Pr(t_{n-k} < t)$

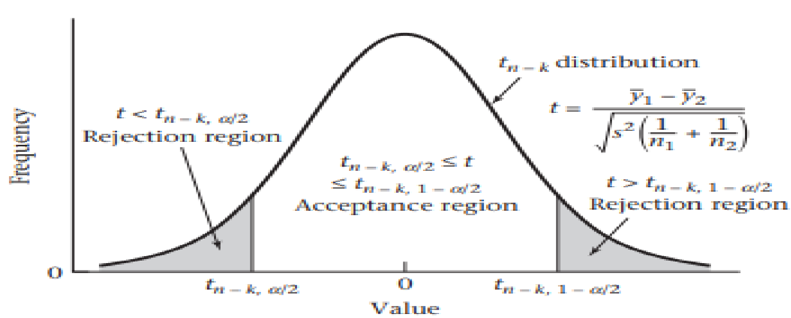
$p = 2 \times$ the area to the right of $t$ under a $t_{n-k}$ distribution if $t \ge 0$
$= 2 \times Pr(t_{n-k} > t)$

(5) A $100\% \times (1 - \alpha)$ CI for $\mu_1 - \mu_2$ is given by

$$\bar{y}_1 - \bar{y}_2 \pm t_{n-k,1-\alpha/2}\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

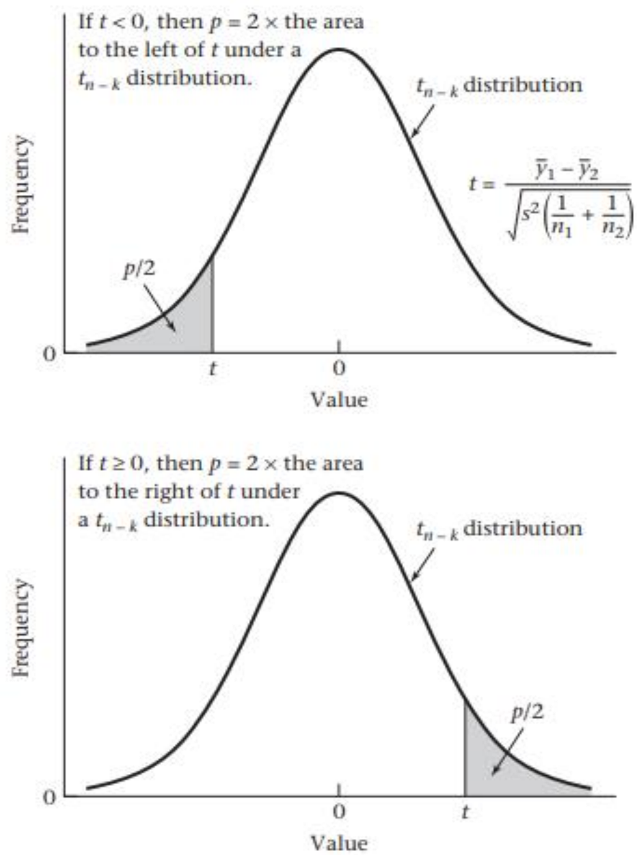
The acceptance and rejection regions for this test are given in Figure 12.4.

**FIGURE 12.4** Acceptance and rejection regions for the $t$ test for the comparison of pairs of groups in one-way ANOVA (LSD approach)



The computation of the exact p-value for the least significant difference (LSD) method is illustrated in Figure 12.5.

**FIGURE 12.5** Computation of the exact $p$-value for the $t$ test for the comparison of pairs of groups in one-way ANOVA (LSD approach)




**EXAMPLE 12.7**

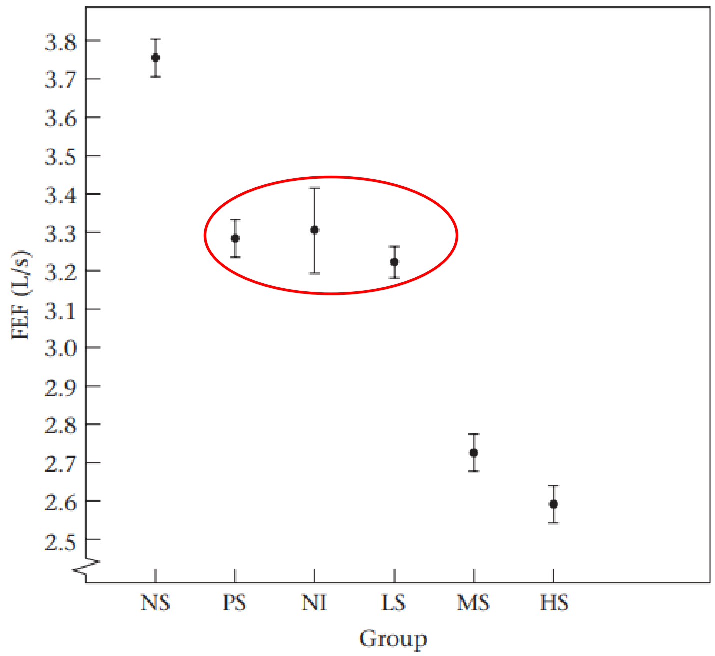
Pulmonary Disease Compare each pair of groups for the FEF data in Table 12.1, and report any significant differences by using the LSD method? Use $\alpha = 0.05$?

Solution

- First, we will plot the (mean ± se) of the FEF values for each of the six groups in Figure 12.6 to obtain some idea of the magnitude of the differences between groups.

- The standard error (se) for an individual group mean is estimated by the formula $se = s/\sqrt{n_i}$, where $s^2 =$ Within MS.

FIGURE 12.6 Mean ± *se* for FEF for each of six smoking groups




Notice that

(1) The nonsmokers have the best pulmonary function; the passive smokers, noninhaling smokers, and light smokers have about the same pulmonary function and are worse off than the nonsmokers; and the moderate and heavy smokers have the poorest pulmonary function.

(2) The standard error bars are wider for the noninhaling smokers than for the other groups because this group has only 50 people compared with 200 for all other groups.

**Question:** Are the observed differences in the figure statistically significant as assessed by the LSD procedure in Equation 12.12? The results are presented in Table 12.4.

**TABLE 12.4** Comparisons of specific pairs of groups for the FEF data in Table 12.1 (on page 552) using the LSD $t$ test approach

| Groups compared | Test statistic | $p$-value |
|---|---|---|
| NS, PS | $t=\dfrac{3.78-3.30}{\sqrt{0.636\left(\dfrac{1}{200}+\dfrac{1}{200}\right)}}=\dfrac{0.48}{0.08}=6.02^{a}$ | < .001 |
| NS, NI | $t=\dfrac{3.78-3.32}{\sqrt{0.636\left(\dfrac{1}{200}+\dfrac{1}{50}\right)}}=\dfrac{0.46}{0.126}=3.65$ | < .001 |
| NS, LS | $t=\dfrac{3.78-3.23}{\sqrt{0.636\left(\dfrac{1}{200}+\dfrac{1}{200}\right)}}=\dfrac{0.55}{0.08}=6.90$ | < .001 |
| NS, MS | $t=\dfrac{3.78-2.73}{0.080}=\dfrac{1.05}{0.08}=13.17$ | < .001 |
| NS, HS | $t=\dfrac{3.78-2.59}{0.080}=\dfrac{1.19}{0.08}=14.92$ | < .001 |
| PS, NI | $t=\dfrac{3.30-3.32}{0.126}=\dfrac{-0.02}{0.126}=-0.16$ | 0.87 |
| PS, LS | $t=\dfrac{3.30-3.23}{0.080}=\dfrac{0.07}{0.08}=0.88$ | 0.38 |
| PS, MS | $t=\dfrac{3.30-2.73}{0.080}=\dfrac{0.57}{0.08}=7.15$ | < .001 |
| PS, HS | $t=\dfrac{3.30-2.59}{0.080}=\dfrac{0.71}{0.08}=8.90$ | < .001 |
| NI, LS | $t=\dfrac{3.32-3.23}{0.126}=\dfrac{0.09}{0.126}=0.71$ | 0.48 |
| NI, MS | $t=\dfrac{3.32-2.73}{0.126}=\dfrac{0.59}{0.126}=4.68$ | < .001 |
| NI, HS | $t=\dfrac{3.32-2.59}{0.126}=\dfrac{0.73}{0.126}=5.79$ | < .001 |
| LS, MS | $t=\dfrac{3.23-2.73}{0.08}=\dfrac{0.50}{0.08}=6.27$ | < .001 |
| LS, HS | $t=\dfrac{3.23-2.59}{0.08}=\dfrac{0.64}{0.08}=8.03$ | < .001 |
| MS, HS | $t=\dfrac{2.73-2.59}{0.08}=\dfrac{0.14}{0.08}=1.76$ | 0.08 |

[a]All test statistics follow a $t_{1044}$ distribution under $H_0$.

Now, the critical value will be obtained from Table 5 in the Appendix, but in this example it will based on the MINITAB program because $df = 1044$ is large, as follows:

$$t_{(n-k\,,\ 1-\alpha/2)} = t_{(1050-6\,,\ 1-0.05/2)} = t_{(1044\,,\ 0.975)} = 1.962$$

## Conclusion

- There are very highly significant differences (t > $t_{(1044\,,\ 0.975)} = 1.962$) between the following pairs of FEF means (*they have different effect on the FEF mean among these groups* (not the same pulmonary function)):
  (1) between the nonsmokers and all other groups:
  [(NS, PS), (NS, NI),(NS, LS), (NS, MS), (NS, HS)].
  (2) between the passive smokers and the moderate and heavy smokers:
  [(PS, MS), (PS, HS)].
  (3) between the noninhalers and the moderate and heavy smokers:
  [(NI, MS), (NI, HS)].
  (4) between the light smokers and the moderate and heavy smokers:
  [(LS, MS), (LS, HS)].

- There are no significant differences (t < $t_{(1044\,,\ 0.975)} = 1.962$) between the following pairs of FEF means (*they have approximately the same effect on FEF mean among these groups* (same pulmonary function)):
  (1) between the passive smokers, noninhalers, and light smokers:
  [(PS, NI), (PS, LS)].
  (2) between the noninhalers and light smokers:
  [(NI, LS)].
  (3) between the moderate and heavy smokers, although there is a trend toward significance with the latter comparison:
  [(MS, HS)].

Thus, these results tend to confirm what Figure 12.6 shows. They are very interesting because they show that the pulmonary function of passive smokers is significantly worse than that of nonsmokers and is essentially the same as that of noninhaling and light smokers (≤ l/2 pack cigarettes per day).

### Notations

- A frequent error in performing the t test in Equation 12.12 when comparing groups 1 and 2 is to use only the sample variances from these two groups rather than from all $k$ groups to estimate $\sigma^2$. If the sample variances from only two groups are used, then different estimates of $\sigma^2$ are obtained for each pair of groups considered, which is not reasonable because all the groups are assumed to have the same underlying variance $\sigma^2$.

- The estimate of $\sigma^2$ obtained by using all $k$ groups will be more accurate than that obtained from using any two groups because the estimate of the variance will be based on more information. This is the principal advantag of a one-way ANOVA rather than by considering each pair of groups separately and performing t tests for two independent samples as given in Equation 8.11 for each pair of samples.
- However, if there is reason to believe that not all groups have the same underlying variance ($\sigma^2$), then the one-way ANOVA should not be performed, and t tests based on pairs of groups should be used instead.

**Exercise:** Study Example 12.5 and Example 12.8?

**Important Notation (*Confidence Interval Method*)**

It can also be interesting to find the $(1-\alpha) \times 100\%$ confidence intervals (CI) for the difference between two group means, say,$(\alpha_i - \alpha_j)$, for example $(\mu_1 - \mu_2)$, as follows:

A $100\% \times (1-\alpha)$ CI for $\mu_1 - \mu_2$ is given by

$$\bar{y}_1 - \bar{y}_2 \pm t_{n-k,1-\alpha/2}\sqrt{s^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

and then the following rule can be used:

(a) If ZERO (0) belongs to ($\in$) the $(1-\alpha) \times 100\%$ confidence intervals (CI), then we conclude that there is no difference in means for the two groups.

(b) If ZERO (0) does not belong to ($\notin$) the $(1-\alpha) \times 100\%$ confidence intervals (CI), then we conclude that there is a difference in means for the two groups.

**Example**

Calculate the 95% confidence intervals for $\mu_{NS} - \mu_{PS}$ and $\mu_{PS} - \mu_{LS}$?

Solution

Step (1): $s^2 = \text{ Within MS} = 0.636$

Step (2): $t_{(1044\,,\ 0.975)} = 1.962$

Step (3): The 95% confidence intervals for $\mu_{NS} - \mu_{PS}$ is given by:

$$\text{CI} = (\bar{y}_{NS} - \bar{y}_{PS}) \pm t_{(n-k\,,\ 1-\alpha/2)} \sqrt{s^2 \left(\frac{1}{n_{NS}} + \frac{1}{n_{PS}}\right)}$$

$$= (3.78 - 3.30) \pm 1.962 \sqrt{0.636 * \left(\frac{1}{200} + \frac{1}{200}\right)} = 0.48 \pm 0.16 = (0.32\,, 0.64)$$

**Conclusion**

We conclude that 0 $\notin \text{CI} = (0.32\,, 0.64)$ which implies that there is a difference in means for the two groups NS and PS.

Step (4): The 95% confidence intervals for $\mu_{PS} - \mu_{LS}$ is given by:

$$\text{CI} = (\bar{y}_{PS} - \bar{y}_{LS}) \pm t_{(n-k\,,\ 1-\alpha/2)} \sqrt{s^2 \left(\frac{1}{n_{PS}} + \frac{1}{n_{LS}}\right)}$$

$$= (3.30 - 3.23) \pm 1.962 \sqrt{0.636 * \left(\frac{1}{200} + \frac{1}{200}\right)} = 0.07 \pm 0.16 = (-0.09\,, 0.23)$$

**Conclusion**

We conclude that 0 $\in \text{CI} = (-0.09\,, 0.23)$ which implies that there is no difference in means for the two groups PS and NS.

---

**Notation**

The typical data used for constructing a one-way ANOVA table would appear as shown in Table below:

| Group | Observations | | | | Totals | Averages |
|---|---|---|---|---|---|---|
| 1 | $y_{11}$ | $y_{12}$ | $\cdots$ | $y_{1n}$ | $y_{1.}$ | $\bar{y}_{1.}$ |
| 2 | $y_{21}$ | $y_{22}$ | $\cdots$ | $y_{2n}$ | $y_{2.}$ | $\bar{y}_{2.}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots\vdots\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $k$ | $y_{k1}$ | $y_{k2}$ | $\cdots$ | $y_{kn}$ | $y_{k.}$ | $\bar{y}_{k.}$ |
| | | Total | | | $y_{..}$ | $\bar{y}_{..}$ |

**Example**

Blood Pressure A researcher wishes to try three different techniques to lower the blood pressure of individuals diagnosed with high blood pressure. The subjects are randomly assigned to three groups $(k = 3)$ as follows:

- First group takes medication (M).
- Second group exercises (E).
- Third group diets (D).

After four weeks, the reduction in each person's blood pressure is recorded. The observed data is given as follows:

| Group $(Reduction\ Technique)$ | Observations | | | | |
|---|---|---|---|---|---|
| | $y_{i1}$ | $y_{i2}$ | $y_{i3}$ | $y_{i4}$ | $y_{i5}$ |
| Medication $(i = 1)$ | 10 | 12 | 9 | 15 | 13 |
| Exercise $(i = 2)$ | 6 | 8 | 3 | 0 | 2 |
| Diet $(i = 3)$ | 5 | 9 | 12 | 8 | 4 |

Answer the following:

(I) At $\alpha = 0.05$, test the claim that there is no difference among the means?

**Solution**

To test this claim, we proceed as follows:

Step (1): Calculate the average $(\bar{y}_{i.})$ and the variance $(s_i^2)$ for each one of the three groups $(i = 1, 2, 3)$ as shown in the table below:

| Group | Observations | | | | | Totals $y_{i.}$ | Averages $\bar{y}_{i.}$ | Variances $s_i^2$ |
|---|---|---|---|---|---|---|---|---|
| | $y_{i1}$ | $y_{i2}$ | $y_{i3}$ | $y_{i4}$ | $y_{i5}$ | | | |
| Medication $(i = 1)$ | 10 | 12 | 9 | 15 | 13 | 59 | 11.8 | 5.7 |
| Exercise $(i = 2)$ | 6 | 8 | 3 | 0 | 2 | 19 | 3.8 | 10.2 |
| Diet $(i = 3)$ | 5 | 9 | 12 | 8 | 4 | 38 | 7.6 | 10.3 |

Step (2): State the hypotheses and identify the claim:

H$_0$: All means of blood pressure reduction observations of the 3 groups is the same, that is, $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (claim).

$versus$

H$_1$: At least two means of blood pressure reduction observations of the 3 groups is not the same, that is, at least one $\alpha_i \neq 0$ ; $i = 1, 2, 3$.

Step (3): Find the critical value as follows:

Since $k = 3$ and $n = \sum_{i=1}^{k=3} n_i = n_1 + n_2 + n_3 = 5 + 5 + 5 = 15$, then:

- $d_1$ ($df$ for numerator)$= k - 1 = 3 - 1 = 2$
- $d_2$ ($df$ for denominator)$= n - k = 15 - 3 = 12$

Thus, the critical value is obtained from the F-table (*Table 8-Percentage points of the F distribution* ($F_{d1,\, d2,\, p}$)) in the Appendix page 882-883 as follows:

$$F_{(d_1,\ d_2,\ p=1-\alpha)} = F_{(k-1,\ n-k,\ 1-\alpha)} = F_{(2,\ 12,\ 1-0.05)}$$
$$= F_{(2,\ 12,\ 0.95)}$$
$$= 3.89$$

**882** APPENDIX Tables

**TABLE 8** Percentage points of the $F$ distribution ($F_{d_1,d_2,p}$)

| *df* for denominator, $d_2$ | $p$ | *df* for numerator, $d_1$: 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | .90 | 3.18 | 2.81 | 2.61 | 2.48 | 2.39 | 2.33 | 2.28 | 2.24 | 2.15 | 2.04 | 1.90 |
| | .95 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.69 | 2.51 | 2.30 |
| | .975 | 6.55 | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.61 | 3.51 | 3.28 | 3.02 | 2.72 |

Step (4): Calculate the test statistic value (*F-value*), using the following procedure:

(a) Compute the Within SS and Between SS for the blood pressure reduction data by using Equation 12.5 as follows:

(1) The sum of the observations across all groups ($y_{..}$) can be calculated as follows:

$$y_{..} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^{k} n_i * \bar{y}_i$$
$$y_{..} = \sum_{i=1}^{3} \sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^{3} n_i * \bar{y}_i$$
$$= n_1 * \bar{y}_1 + n_2 * \bar{y}_2 + n_3 * \bar{y}_3$$

$$= (5)(11.8) + (5)(3.8) + (5)(7.6)$$
$$= 116$$

(2) The Between Sum of Squares (Between SS) can be calculated as follows:

$$\text{Between SS } = \sum_{i=1}^{k=3} n_i\, \bar{y}_i^2 - \frac{y_{..}^2}{n}$$
$$= [(5)(11.8)^2 + (5)(3.8)^2 + (5)(7.6)^2] - \frac{(116)^2}{15}$$
$$= 1057.2 - 897.067$$
$$= 160.133$$

(3) The Within Sum of Squares (Within SS) can be calculated as follows:

$$\text{Within SS } = \sum_{i=1}^{k=3} (n_i - 1)\, s_i^2$$
$$= (n_1 - 1)\, s_1^2 + (n_2 - 1) s_2^2 + (n_3 - 1)\, s_3^2$$
$$= (4)(5.7) + (4)(10.2) + (4)(10.3)$$
$$= 104.8$$

(b) Compute the Within MS and Between MS for the blood pressure reduction data as follows:

(1) Between SS = 160.133, then:

Between MS = Between SS $/(k - 1)$
= 160.133 / 2
= 80.0665

(2) Within SS = 104.8, then:

Within MS = Within SS $/(n - k)$
= 104.8 / (15 - 3)
= 104.8 / 12
= 8.7333

(c) The test statistic value (calculated *F-value*) is obtained as follows:

F = Between MS / Within MS
= 80.0665 / 8.7333
= 9.17 $\sim F_{(k-1,\ n-k,\ 1-\alpha)} = F_{(2,\ 12,\ 0.95)}$ under $H_0$.

(d) The exact $p - value$ (given by the area to the right of F under an $F_{(k-1,\ n-k,\ 1-\alpha)}$ distribution) can be calculated as follows:

$$p - value = P\left(F_{(k-1,\ n-k,\ 1-\alpha)} > F\right)$$
$$= P\left(F_{(2,\ 12,\ 0.95)} > 9.17\right)$$
$$= 0.004 < \alpha = 0.05$$

(d) One-Way ANOVA Table

The results obtained in (a) – (c) are displayed in an ANOVA table (*One-Way ANOVA Table*) which is shown below:

One-Way ANOVA Table

<table>
<tr><th>Source of Variation</th><th>SS</th><th>df</th><th>MS</th><th>F-value</th><th>p-value</th></tr>
<tr><td>Between</td><td>160.133</td><td>2</td><td>80.0665</td><td rowspan="3">9.17</td><td rowspan="3">0.004</td></tr>
<tr><td>Within</td><td>104.8</td><td>12</td><td>8.7333</td></tr>
<tr><td>Total</td><td>264.933</td><td>14</td><td></td></tr>
</table>

Step (5): Make the decision. The decision is to reject the null hypothesis ($H_0$), since we get $F - value = 9.7 > F_{(2,\ 12,\ 0.95)} = 3.89$.

Step (6): Conclusion and summarizes the results. There is enough evidence to reject the claim and conclude that at least one mean is different from the others.

(II) At $\alpha = 0.05$, use the least significant difference (LSD) method to determine specific differences between blood pressure reduction techniques?

**Solution**

Science the decision in part (I) indicates that a difference exists between the means of the blood pressure reduction techniques (*because we reject $H_0$*), then we will perform the least significant difference (LSD) method to isolate the specific difference.

Step (1): Critical Value

The critical value will be obtained from Table 5 in the Appendix based on degrees of freedom $df = n - k = 15 - 3 = 12$, as follows:

$$t_{(n-k\,,\ 1-\alpha/2)} = t_{(15-3\,,\ 1-0.05/2)} = t_{(12\,,\ 0.975)} = 2.179$$

Step (2): $s^2 = \text{Within MS} = 8.7333$

Step (3): The value of the test statistic ($t$) for the all pairs of compared groups is calculated as follows:

(a) Groups Compared - Medication (M) and Exercise (E):

Hypothesis: H$_0$: $\alpha_M = \alpha_E \ \ versus$ H$_1$: $\alpha_M \neq \alpha_E$

$$t = \frac{\bar{y}_M - \bar{y}_E}{\sqrt{s^2\left(\frac{1}{n_M} + \frac{1}{n_E}\right)}} = \frac{11.8 - 3.8}{\sqrt{8.7333\left(\frac{1}{5} + \frac{1}{5}\right)}} = \frac{8}{1.869} = 4.280$$

(b) Groups Compared - Medication (M) and Diet (D):

Hypothesis: H$_0$: $\alpha_M = \alpha_D \ \ versus$ H$_1$: $\alpha_M \neq \alpha_D$

$$t = \frac{\bar{y}_M - \bar{y}_D}{\sqrt{s^2\left(\frac{1}{n_M} + \frac{1}{n_E}\right)}} = \frac{11.8 - 7.6}{\sqrt{8.7333\left(\frac{1}{5} + \frac{1}{5}\right)}} = \frac{4.2}{1.869} = 2.247$$

(c) Groups Compared - Exercise (E) and Diet (D):

Hypothesis: H$_0$: $\alpha_E = \alpha_D \ \ versus$ H$_1$: $\alpha_E \neq \alpha_D$

$$t = \frac{\bar{y}_E - \bar{y}_D}{\sqrt{s^2\left(\frac{1}{n_M} + \frac{1}{n_E}\right)}} = \frac{3.8 - 7.6}{\sqrt{8.7333\left(\frac{1}{5} + \frac{1}{5}\right)}} = \frac{-3.8}{1.869} = -2.033$$

Therefore, the results of the comparisons using the LSD method are presented in the following table:

| **Groups Compared** | **Test Statistic Value** | **Critical Value** | **Decision** |
|---|---|---|---|
| Medication (M) , Exercise (E) | 4.280 | 2.179 | Reject H$_0$ |
| Medication (M) , Diet (D) | 2.247 | 2.179 | Reject H$_0$ |
| Exercise (E) , Diet (D) | – 2.033 | 2.179 | Accept (Do Not Reject) H$_0$ |

Step (4): Conclusion

There are no significant differences (t = -2.033 < $t_{(12,\ 0.975)} = 2.179$) between the Exercise (E) and Diet (D) means. Both techniques, Exercise and Diet, have approximately the same effect on lowering the blood pressure of individuals diagnosed with high blood pressure.

(III) Find a 95% confidence intervals for the difference between the mean blood pressure reduction for all techniques $\mu_M - \mu_E$ , $\mu_M - \mu_D$ and $\mu_E - \mu_D$?

**Solution**

The $(1-\alpha) \times 100\%$ confidence intervals (CI) for the difference between two group means, say,$(\alpha_i - \alpha_j)$, for example $(\mu_1 - \mu_2)$, can be obtained as follows:

A $100\% \times (1-\alpha)$ CI for $\mu_1 - \mu_2$ is given by

$$\bar{y}_1 - \bar{y}_2 \pm t_{n-k,1-\alpha/2} \sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Step (1): $s^2 =$ Within MS $= 8.7333$

Step (2): $t_{(12\,,\ 0.975)} = 2.179$

Step (3): The 95% confidence interval for $\mu_M - \mu_E$ is given by:

$$\text{CI} = (\bar{y}_M - \bar{y}_E) \pm t_{(n-k\,,\ 1-\alpha/2)} \sqrt{s^2\left(\frac{1}{n_M} + \frac{1}{n_E}\right)}$$

$$= (11.8 - 3.8) \pm 2.179\sqrt{8.7333 * \left(\tfrac{1}{5} + \tfrac{1}{5}\right)} \;= 8 \pm 4.073 = (3.93\,, 12.07)$$

Conclusion

We conclude that 0 $\notin$ CI $= (3.93\,, 12.07)$ which implies that there is a difference in means for the two techniques, Medication (M) and Exercise (E), on lowering the blood pressure of individuals diagnosed with high blood pressure.

Step (4): The 95% confidence interval for $\mu_M - \mu_D$ is given by:

$$\text{CI} = (\bar{y}_M - \bar{y}_D) \pm t_{(n-k\,,\ 1-\alpha/2)} \sqrt{s^2\left(\frac{1}{n_M} + \frac{1}{n_D}\right)}$$

$$= (11.8 - 7.6) \pm 2.179\sqrt{8.7333 * \left(\tfrac{1}{5} + \tfrac{1}{5}\right)} \;= 4.2 \pm 4.073 = (0.13\,, 8.27)$$

Conclusion

We conclude that 0 $\notin$ CI $= (0.13\,, 8.27)$ which implies that there is a difference in means for the two techniques, Medication (M) and Diet (D), on lowering the blood pressure of individuals diagnosed with high blood pressure.

Step (5): The 95% confidence interval for $\mu_E - \mu_D$ is given by:

$$\text{CI} = (\bar{y}_E - \bar{y}_D) \pm t_{(n-k\,,\;1-\alpha/2)} \sqrt{s^2\left(\frac{1}{n_E} + \frac{1}{n_D}\right)}$$

$$= (3.8 - 7.6) \pm 2.179\sqrt{8.7333 * \left(\tfrac{1}{5} + \tfrac{1}{5}\right)} = -3.8 \pm 4.073 = (-7.87\,, 0.27)$$

Conclusion

We conclude that $0 \in \text{CI} = (-7.87\,, 0.27)$ which implies that there is no difference in means for the two techniques, Exercise (E) and Diet (D), on lowering the blood pressure of individuals diagnosed with high blood pressure.

Therefore, the results of the 95% confidence intervals for the diferences between the mean blood pressure reduction for all reduction techniques $\mu_M - \mu_E$ , $\mu_M - \mu_D$ and $\mu_E - \mu_D$ are presented in the following table:

| **Mean Differences** | **95% Confidence Interval** | | **Includes Zero** | **Decision** |
|---|---|---|---|---|
| | Lower Limit | Upper Limit | | |
| $\mu_M - \mu_E$ | $3.93$ | $12.07$ | No | Reject $H_0$ |
| $\mu_M - \mu_D$ | $0.13$ | $8.27$ | No | Reject $H_0$ |
| $\mu_E - \mu_D$ | $-7.87$ | $0.27$ | Yes | Accept (Do Not Reject) $H_0$ |